A LIGHTFACE ANALYSIS OF THE DIFFERENTIABILITY RANK

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Abstract. We examine the computable part of the differentiability hierarchy defined by Kechris and Woodin. In that hierarchy, the rank of a differentiable function is an ordinal less than ω_1 which measures how complex it is to verify differentiability for that function. We show that for each recursive ordinal $\alpha > 0$, the set of Turing indices of $C[0,1]$ functions that are differentiable with rank at most α is $\Pi_{2\alpha+1}$-complete. This result is expressed in the notation of Ash and Knight.

1. Introduction

The set of differentiable $C[0,1]$ functions is not Borel, but it can be represented hierarchically as an increasing union of Borel sets. Three hierarchies for the differentiable functions have been proposed (see the work of Ki [Ki97] for a summary). Each hierarchy is defined using an *ordinal rank*, a mapping from differentiable functions to countable ordinals, whose range is unbounded below ω_1. Here we focus on Kechris and Woodin's differentiability rank [KW86], denoted $|\cdot|_{KW}$. It decomposes the set $\mathcal{D}$ of differentiable $C[0,1]$ functions as

$$\mathcal{D} = \bigcup_{\alpha<\omega_1} \{f : |f|_{KW} < \alpha\}$$

where each constituent of the union is Borel.

Our contribution is a finer-grained, recursion-theoretic analysis of this hierarchy. The lightface situation mirrors the boldface situation in many ways. We begin with the observation (a corollary of the work in [KW86]) that the set D of integer codes for computable differentiable $C[0,1]$ functions is a Π^1_1-complete set, and it decomposes as

$$D = \bigcup_{\alpha<\omega_1^{CK}} \{c : c \text{ codes } f \text{ with } |f|_{KW} < \alpha\}$$

where each constituent of the union is hyperarithmetic. Our results pinpoint the exact location of each constituent set in the hyperarithmetical hierarchy.

Theorem 4.9. *For each nonzero $\alpha < \omega_1^{CK}$, the set*

$$\{c : c \text{ codes } f \text{ with } |f|_{KW} < \alpha + 1\}$$

is $\Pi_{2\alpha+1}$-complete.

Here and throughout we use the notational convention of Ash and Knight [AK00] for a Σ_α set, discussed in Section 2.2. We also analyze the limit case:

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Theorem 4.10. *For each limit $\lambda < \omega_1^{CK}$, the set $\{c : c \text{ codes } f \text{ with } |f|_{KW} < \lambda\}$ is Σ_λ-complete.*

The study of differentiation through the lens of computable analysis has typically involved restricting attention to the continuously differentiable functions. The definition of a computable function proposed by Grzegorczyk and Lacombe, and further developed by Pour-El and Richards and others (see [Grz57], [Lac55], [PER89]), has no notion of computability for a discontinuous function. Therefore, restricting differentiation to the continuously differentiable functions is a strategy for making questions such as "Is differentiation computable?" meaningful. The fact that $f \mapsto f'$ is not computable was first demonstrated by Myhill [Myh71], who constructed a computable function whose continuous derivative is not computable.

At the other end of the spectrum, computable functions that are not everywhere differentiable have been studied. Brattka, Miller and Nies (to appear) have used randomness notions to characterize the points at which all computable almost everywhere differentiable functions must be differentiable. However, as far as the author is aware, the everywhere differentiable functions with discontinuous derivatives have not yet been studied in the setting of computable analysis.

Previously, Cenzer and Remmel [CR04] showed that $\{e : f_e \text{ is continuously differentiable}\}$ is Π^0_3-complete, which is the same as the $\alpha = 1$ case of our Theorem 4.9. They also showed that $\{e : f_e \text{ is continuously differentiable with } f'_e \text{ computable}\}$ is Σ^0_3-complete. Again, only continuously differentiable functions were considered. By contrast, our aim is to provide a clearer picture of the structure of the unrestricted set of everywhere differentiable functions.

In Section 2 we review the basic facts about computable $C[0,1]$ functions, the ordinals below ω_1^{CK}, and Σ_α sets. Then we introduce Kechris and Woodin's differentiability rank, and present what is known about D. In Section 3 we redefine the differentiability rank in a more computationally convenient way, and use this definition to demonstrate $\{c : c \text{ codes } f \text{ with } |f|_{KW} < \alpha + 1\}$ is $\Pi_{2\alpha+1}$. The meat of the paper is in Section 4, where we address the question of completeness to prove both theorems above.

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2. Preliminaries

This section provides background, essential definitions, methods previously used to construct functions of different ranks, and corollaries that are straightforward effectivizations of arguments in the literature. In section 2.1 we establish some notation and review the basic facts about computable $C[0,1]$ functions. In Section 2.2 we introduce the recursive ordinals and use them to define Σ_α-completeness. In Section 2.3 we define Kechris and Woodin's differentiability rank. In Section 2.4 we familiarize the reader with the building blocks used in [KW86] to construct functions of arbitrary rank, as these essential elements are taken for granted in what follows. In Section 2.5 we establish more notation that is used throughout the paper. Finally, in Section 2.6 we present some necessary facts about computable differentiable functions that can be obtained by effectivizing existing work.

2.1. **Basic notions and encoding $C[0,1]$ functions.** We use ϕ_e to denote to the eth Turing functional, and W_e refers to the domain of ϕ_e. We identify subsets $X \subseteq \mathbb{N}$ with their characteristic functions $X \in 2^\omega$. The jump of $X \in 2^\omega$ is written X', and the nth jump of X is written $X^{(n)}$. Turing reducibility is denoted by $\leq_T$ and one-reducibility by $\leq_1$. We use $\langle n_1, \dots, n_k \rangle$ to denote a single integer which represents the tuple $(n_1, \dots, n_k)$ according to some standard computable encoding. If $\tau = \langle m_1, \dots, m_r \rangle$ and $\sigma = \langle n_1, \dots, n_k \rangle$, let $\tau^\frown\sigma$ denote $\langle m_1, \dots, m_r, n_1, \dots, n_k \rangle$. If $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a tree, let T_n denote $\{\sigma : \langle n \rangle^\frown\sigma \in T\}$, the nth subtree of T. If T is well-founded, $|T|$ denotes its rank.

We identify the computable functions with the computable subsets of $\mathbb{N}$ that encode those functions. Following [KW86], all our functions are real-valued with domain $[0,1]$. For the encoding we use Simpson's definition from [Sim09] because this encoding makes it straightforward to determine the degree of unsolvability of various statements. For example, we will observe that "ϕ_e encodes a computable $C[0,1]$ function" is Π_2. However, the exact details of the Simpson encoding are not needed beyond this section, and any of the many equivalent definitions for a computable real-valued function can be safely substituted.

In the following definition, $(a,r)\Phi(b,s)$ is shorthand for $\exists n((n,a,r,b,s,) \in \Phi)$, and $(a,r) < (a',r')$ means that $|a-a'|+r' < r$. The idea is that $(a,r)\Phi(b,s)$ should mean that $f(B(a,r)) \subseteq \overline{B(b,s)}$.

Definition 2.1. *A code for a continuous partial function f from $[0,1]$ to $\mathbb{R}$ is a set of quintuples $\Phi \subseteq \mathbb{N} \times \mathbb{Q} \cap [0,1] \times \mathbb{Q}^+ \times \mathbb{Q} \times \mathbb{Q}^+$ which satisfies:*

(1) *if $(a,r)\Phi(b,s)$ and $(a,r)\Phi(b',s')$ then $|b-b'| \leq s+s'$*
(2) *if $(a,r)\Phi(b,s)$ and $(a',r') < (a,r)$, then $(a',r')\Phi(b,s)$*
(3) *if $(a,r)\Phi(b,s)$ and $(b,s) < (b',s')$, then $(a,r)\Phi(b',s')$.*

This set Φ is coded as a subset of $\mathbb{N}$ using the standard encoding. Some important facts can be seen from this definition. Firstly, it is Π_2 to check whether a given code $X \subseteq \mathbb{N}$ satisfies the above properties. Secondly, the codes satisfying the above might not represent total functions. That is, for some points x in $[0,1]$ and some ε there may not be an a,r,b such that $|x-a| < r$ and $(a,r)\Phi(b,\varepsilon)$. However if the code does represent a total function then, by the compactness of $[0,1]$, for each ε there is a finite set $\{(a_i,r_i,b_i,s_i)\}_{i<p}$ such that the (a_i,r_i) cover $[0,1]$ and for each i, $s_i \leq \varepsilon$ and $(a_i,r_i)\Phi(b_i,s_i)$. Therefore, "ϕ_e encodes a $C[0,1]$ function" is a Π_2 statement: ϕ_e is total, and the corresponding code satisfies Definition 2.1, and for all ε there is a finite cover as described above. Let f_e denote the $C[0,1]$ function encoded by ϕ_e. Note that any function encoded using this convention is, by necessity, continuous.

If f is any computable $C[0,1]$ function and z and x any rational numbers, the statement $f(x) > z$ is Σ_1, because $f(x) > z$ if and only if there are δ, b and ε such that $(x,\delta)\Phi(b,\varepsilon)$ and $b - \varepsilon > z$.

We will also freely make use of the fact that addition, multiplication, division, and composition of computable functions are computable. For details we refer the reader to [Sim09].

2.2. **Kleene's $\mathcal{O}$ and the notion of a Σ_α-complete set.** Kleene's $\mathcal{O}$ is a way of encoding the recursive ordinals as natural numbers. First one defines a relation $<_\mathcal{O}$ on $\mathbb{N}$ as the least relation closed under the following properties:

(1) $1 <_\mathcal{O} 2$.

(2) If $a <_{\mathcal{O}} b$ then $b <_{\mathcal{O}} 2^b$.
(3) If $\phi_e(n)$ is total and $\phi_e(n) <_{\mathcal{O}} \phi_e(n+1)$ for all n, then $\phi_e(n) <_{\mathcal{O}} 3 \cdot 5^e$ for all n.
(4) If $a <_{\mathcal{O}} b$ and $b <_{\mathcal{O}} c$ then $a <_{\mathcal{O}} c$.

The field of this relation is called Kleene's $\mathcal{O}$. One can show that $\mathcal{O}$ is a Π^1_1-complete set, that $<_{\mathcal{O}}$ is well-founded, and for each $a \in \mathcal{O}$, the set $\{b : b <_{\mathcal{O}} a\}$ is well ordered and computably enumerable. (See [Sac90] for details). Therefore, for each $a \in \mathcal{O}$ there is a well-defined ordinal $|a|_{\mathcal{O}} = \mathrm{ot}(\{b : b <_{\mathcal{O}} a\})$. In this situation a is called an *ordinal notation* for $|a|_{\mathcal{O}}$. If an ordinal has an ordinal notation in $\mathcal{O}$, it is called a *constructive ordinal*. Note that there are infinitely many ordinal notations corresponding to each constructive ordinal $\alpha \geq \omega$. There are only countably many constructive ordinals and these form an initial segment of the ordinals. The least nonconstructive ordinal is called ω_1^{CK}, "the ω_1 of Church and Kleene".

We will use the fact that it is computable to add ordinal notations in a way that is consistent with their corresponding ordinals.

The constructive ordinals have an important equivalent characterization. They are exactly the ranks of the recursive well-founded relations. This will be used to establish that the differentiability ranks of the computable functions are the constructive ordinals.

We recall the arithmetical hierarchy for $n < \omega$. A set X is said to be Σ_n (respectively Π_n) if $X \leq_1 \emptyset^{(n)}$ (respectively $\overline{\emptyset^{(n)}}$), and X is Σ_n-complete if $X \equiv_1 \emptyset^{(n)}$ (and similarly for Π_n-completeness).

The ordinal notations provide a natural way to extend the notion of the Turing jump through the ordinals less than ω_1^{CK}, giving rise to the hyperarithmetical hierarchy. Define $H_1 = \emptyset$, $H_{2^b} = (H_b)'$, and $H_{3\cdot 5^e} = \{\langle x, n\rangle : x \in H_{\phi_e(n)}\}$. Spector [Spe55] showed that if $|a|_{\mathcal{O}} = |b|_{\mathcal{O}}$, then $H_a \equiv_T H_b$. Therefore, $H_{2^a} \equiv_1 H_{2^b}$, and thus there is a well-defined notion of one-reducibility and completeness at the successor levels. We define the notions of Σ_α and Π_α for infinite ordinals following [AK00]:

Definition 2.2. *Let $\alpha < \omega_1^{CK}$ be an infinite ordinal and let $X \in 2^\omega$. Then X is Σ_α if $X \leq_1 H_{2^a}$ for any a such that $|a|_{\mathcal{O}} = \alpha$, and X is Σ_α-complete if $X \equiv_1 H_{2^a}$ for any such a. The Π_α and Π_α-complete sets are defined similarly.*

Note that using this definition, $(\emptyset^{(\omega)})'$ is a Σ_ω-complete set. There is a conflicting notational convention, found in [Soa87, pg. 259], in which $(\emptyset^{(\omega)})'$ is classified $\Sigma_{\omega+1}$-complete, and the symbol Σ_ω is not defined. We prefer the notation of [AK00] because it is more consonant with the definition of the rank function. As will be seen, to determine whether the core rank-ascertaining process terminates at a limit stage, it is necessary to use a quantification over the results of the previous stages, not merely a unified presentation of them.

We fix a particular (but arbitrary) path $\mathcal{P}$ through $\mathcal{O}$ and define $\emptyset^{(\alpha)}$ for each $\alpha < \omega_1^{CK}$ by $\emptyset^{(\alpha)} = H_a$, where a is the unique $a \in \mathcal{P}$ such that $|a|_{\mathcal{O}} = \alpha$. (We call $\mathcal{P}$ a path through $\mathcal{O}$ if $\mathcal{P} \subseteq \mathcal{O}$ is $<_{\mathcal{O}}$-linearly ordered and contains an ordinal notation for each $\alpha < \omega_1^{CK}$.)

Because $\emptyset^{(\alpha+1)}$ is the canonical Σ_α-complete set when $\alpha > \omega$, we follow [GMS11] in defining

$$\emptyset_{(\alpha)} = \begin{cases} \emptyset^{(\alpha)} & \text{if } \alpha < \omega \\ \emptyset^{(\alpha+1)} & \text{if } \alpha \geq \omega \end{cases}$$

so that $\emptyset_{(\alpha)}$ is always the canonical Σ_α-complete set. In addition, we identify α with the relevant ordinal notation, which in this paper is the notation $a \in P$ such that $H_a = \emptyset_{(\alpha)}$. (Thus infinite α are identified with the a such that $|a|_{\mathcal{O}} = \alpha + 1$). This choice greatly simplifies the presentation in Section 4 by removing the need to explicitly and constantly deal with the non-uniformity between the finite and the infinite discussed here.

As we are in the business of establishing the Π_α-completeness of various sets, we will construct reductions to and from $\emptyset_{(\alpha)}$ for various values of α. All of our reductions will be either to some $\emptyset_{(\alpha)}$ or to index sets. Since all sets of these kinds permit padding, it will suffice to find many-one reductions, and this is what we do. We use the technique of effective transfinite recursion which is described in detail in [Sac90]. For our purposes it can be stated as follows:

Theorem 2.1. *Let $I : \omega \to \omega$ be a recursive function, and suppose for all $e \in \mathbb{N}$ and all $x \in \mathcal{P}$, if $\phi_e(y)$ is defined for all $y \in \mathcal{P}$ such that $y <_{\mathcal{O}} x$, then $\phi_{I(e)}(x)$ is defined. Then for some c, $\phi_c(x)$ is defined for all $x \in \mathcal{P}$, and $\phi_c(x) = \phi_{I(c)}(x)$ for any x on which either converges.*

When we use this technique, the function I will be defined only implicitly.

2.3. **Kechris and Woodin's differentiability rank.** Kechris and Woodin [KW86] define a rank on differentiable $C[0,1]$ functions as follows. Let $\Delta_f(x, y)$ denote the secant slope

$$\Delta_f(x, y) = \frac{f(x) - f(y)}{x - y}.$$

They define a "derivative" operation, which is given below. This operation starts with a closed set of points P and removes from it some points at which f seems to be differentiable. A point x is removed if the oscillation of f' near x is no more than the given ε.

Definition 2.3. *Given a closed set P, a function $f \in C[0,1]$ and $\varepsilon > 0$,*

$$\begin{aligned} P'_{f,\varepsilon} = \{x \in P : \forall\delta > 0 \exists p < q, r < s \in B(x,\delta) \cap [0,1] \\ \textit{with } [p,q] \cap [r,s] \cap P \neq \emptyset \textit{ and } |\Delta_f(p,q) - \Delta_f(r,s)| \geq \varepsilon\} \end{aligned}$$

where all the quantifiers range over rational numbers.

If P is closed, then P' is closed as well, so for each $f \in C[0,1]$ and each $\varepsilon > 0$ one defines the following inductive hierarchy:

$$\begin{aligned} P^0_{f,\varepsilon} &= [0,1] \\ P^{\alpha+1}_{f,\varepsilon} &= (P^\alpha_{f,\varepsilon})'_{f,\varepsilon} \\ P^\lambda_{f,\varepsilon} &= \cap_{\alpha<\lambda} P^\alpha_{f,\varepsilon} \text{ for a limit } \lambda \end{aligned}$$

Kechris and Woodin showed that for any $f \in C[0,1]$, f is differentiable if and only if $\forall n \exists \alpha < \omega_1 (P^\alpha_{f,1/n} = \emptyset)$. Considering the supremum of all such α, they make the following definition:

Definition 2.4. *For each differentiable $f \in C[0,1]$, define $|f|_{KW}$ to be the least ordinal α such that $\forall\varepsilon P^\alpha_{f,\varepsilon} = \emptyset$.*

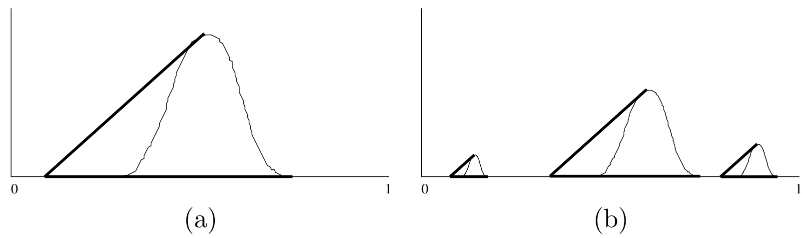


FIGURE 1. (a) A continuously differentiable bump with one secant of slope zero and one secant of positive slope. (b) Resized copies of this bump with proportions preserved.

For example, if f is any continuously differentiable function, then $|f|_{KW} = 1$, the least possible. To see that $P^1_{f,\varepsilon} = \emptyset$ for any such f and any ε, let δ be s.t. $|f'(z) - f'(y)| < \varepsilon$ whenever $|z - y| < \delta$. Then for any x and any $p < q, r < s \in B(x, \delta/2)$, the Mean Value Theorem provides $y \in [p, q]$ and $z \in [r, s]$ such that $f'(y) = \Delta_f(p, q)$ and $f'(z) = \Delta_f(r, s)$, so $|\Delta_f(p, q) - \Delta_f(r, s)| < \varepsilon$ and $x \notin P^1_{f,\varepsilon}$. A common example of a differentiable function whose derivative is not continuous is $x^2 \sin(1/x)$, and this function has differentiability rank 2.

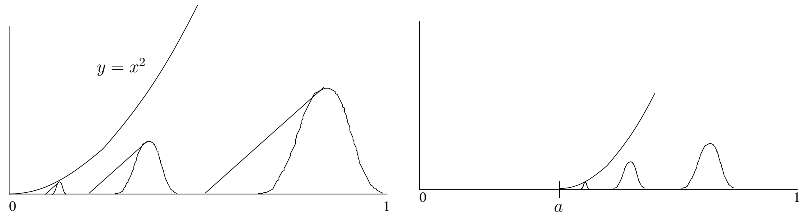
2.4. **Basic building blocks.** Kechris and Woodin show that for each ordinal α, there is a function with rank α, and in order to show this they construct an explicit f with that rank. This section gives a summary of the building blocks that they used to produce an example of a function living at each level of their hierarchy. We will use the same building blocks in a more complicated construction in Section 4.

The most natural way of constructing a function while controlling its rank is to build it up recursively from smaller pieces. Our basic building block is a simple continuously differentiable bump (Figure 1).

Observe a certain pair of secants made by the existence of the bump, one with slope zero and one with positive slope. We build functions out of resized copies of this same bump, always preserving the proportions to keep the corresponding slopes uniform. In Definition 2.3 there is a free parameter ε, and one compares various secants to see if their slope difference is at least ε. Therefore, by choosing a single sufficiently small value for ε, all the secant pairs induced by the bumps are made visible for the purposes of the rank-ascertaining process. We will sometimes refer to ε as the *oscillation sensitivity* because it sets the threshold above which oscillations in the value of the derivative matter.

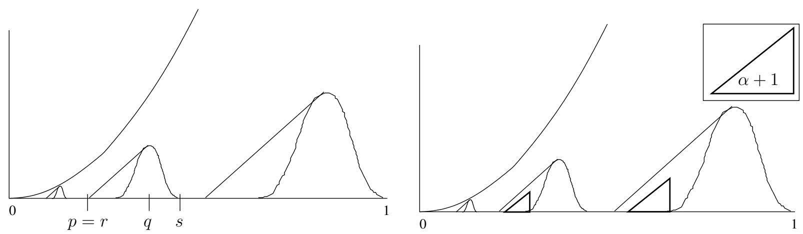
A simple rank 2 function is pictured in Figure 2. To keep 0 from being removed at the first iteration, we put a bump (and thus a disagreeing pair of secants) in every neighborhood of 0. To ensure the function remains differentiable at 0 despite all the oscillation, we make the bumps small enough to fit inside an envelope of x^2. The resulting rank 2 function can itself be proportionally shrunk and used as a building block in functions of larger rank.

The reason 0 is removed at the second iteration, despite infinitely many pairs of disagreeing secants, is that P^1 contains no points which lie in the intersection $[p, q] \cap [r, s]$, where p, q, r, s are the endpoints of the intervals defining the disagreeing secant pair as shown in Figure 3. But if we have a rank $\alpha + 1$ function to use as a building block (the rank must be a successor for reasons discussed below), we can



(a) (b)

FIGURE 2. (a) A simple differentiable function of rank 2. (b) A shifted and resized copy of this function, which fits in a small neighborhood of the point a and keeps a alive through the first iteration.



(a) (b)

FIGURE 3. (a) Points p, q, r, s as used in Definition 2.3. (b) A differentiable function of rank $\alpha + 2$. The triangle represents a function of rank $\alpha + 1$.

make 0 survive the $(\alpha+1)$st iteration. By putting a shrunken copy of our rank $\alpha+1$ function in $[p, q] \cap [r, s]$ as shown in Figure 3, we construct a function of rank $\alpha+2$. We say that we have put the rank $\alpha + 1$ function in the *shadow* of each bump. In fact, it would suffice to put a rank $\alpha + 1$ function in the shadow of infinitely many of the bumps, and this is done later in the paper.

Next we describe how to make functions of rank $\lambda + 1$ and rank λ, where λ is a limit ordinal. We say that an oscillation sensitivity ε *witnesses* the rank of a function f if $|f|_{KW} = \alpha$ and $P_\varepsilon^\beta \neq \emptyset$ for all $\beta < \alpha$. Note that if a function has successor rank, there is always an ε that witnesses this, but if the function has limit rank, there cannot be a witness.

Suppose we have a sequence of functions, with ranks cofinal in λ, whose ranks are all witnessed at a uniform sensitivity ε. As shown in Figure 4, a function of rank $\lambda + 1$ can be made by putting proportionally shrunken copies of functions of increasing rank in each neighborhood of 0. The rank of the resulting function is witnessed by the same ε.

By recursively applying the $\alpha+2$ step and the $\lambda+1$ step, we can build functions of any successor rank. To make a function of rank λ, we must start with a sequence of functions with uniformly bounded derivatives, whose ranks are cofinal in λ. Because the derivatives are uniformly bounded, their possible secant slope differences are also uniformly bounded by the Mean Value Theorem. Again we use shrunken copies of functions from the sequence, but in addition to shrinking the nth function

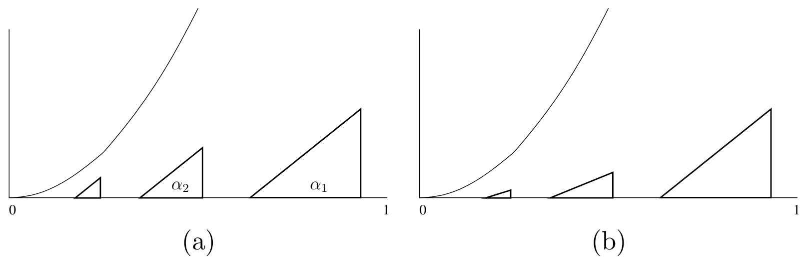


FIGURE 4. (a) A function of rank $\lambda+1$ for λ a limit ordinal. (b) A function of rank λ.

proportionally, we also scale it vertically by a factor of $\frac{1}{n}$. In the resulting function, as x approaches 0 the nearby secant slope differences approach zero, which has the effect of ensuring that 0 is removed at the first iteration no matter what the oscillation sensitivity.

Functions whose ranks are limit ordinals do not make good building blocks for more complicated functions because there is no ε that witnesses their rank. If we construct a rank $\lambda+1$ function f, there needs to be a ε such that $P^{\lambda}_{\varepsilon,f} \neq \emptyset$. If we used a rank λ function g as a building block, then by compactness there would have to be some $\beta < \lambda$ such that $P^{\beta}_{\varepsilon,g} = \emptyset$. So a function of rank β would have been equally unhelpful. That explains why, in our construction of the rank $\alpha+2$ function above, we needed to use a function with successor rank $\alpha+1$ as a building block.

2.5. **Notation.** The following notations are used throughout.

Definition 2.5. *For each ordinal α, let D_α denote the set of all indices e such that $f_e \in C[0,1]$ is differentiable with $|f_e|_{KW} < \alpha$. Define $D = \cup_{\alpha<\omega_1} D_\alpha$.*

For any function $f \in C[0,1]$, we write $f[a,b]$ to denote the function which is identically 0 outside of $[a,b]$, and for $x \in [a,b]$, $f[a,b](x) = (b-a)f(\frac{x-a}{b-a})$. Note that if f is continuous and $f(0) = f(1) = 0$, then $f[a,b]$ is continuous; it is computable when f, a, and b are and differentiable when f is differentiable and $f'(0) = f'(1) = 0$.

Similarly, for any real number $c \in [0,1]$ and any interval $[a,b]$, let $c[a,b] = a+c(b-a)$. This notation comes in handy when talking about scaled down versions of functions, because $(b-a)f(c) = f[a,b](c[a,b])$. Also, this scaling preserves a function's proportions $(f[a,b]'(c[a,b]) = (b-a)f'(c)\frac{1}{b-a} = f'(c))$, so $||f'|| = ||f[a,b]'||$ for any interval $[a,b]$.

2.6. **Facts about** D**.** In section 2.4, we described the major components of Kechris and Woodin's construction of an explicit f with $|f|_{KW} = \alpha$ for each α. When $\alpha < \omega_1^{CK}$, their construction by transfinite recursion easily effectivizes. Therefore their argument also shows that for each constructive α, there is a computable differentiable f with rank α.

On the other hand, every computable differentiable function has constructive rank. This follows from work in the same paper by Kechris and Woodin.

Definition 2.6. *Let $\mathcal{D}$ denote the set of differentiable functions in $C[0,1]$.*

Definition 2.7. *For each function $f \in C[0,1]$ and each $\varepsilon \in \mathbb{Q}^+$, define a tree S_f^ε on $A = \{\langle p, q\rangle \ : 0 \leq p < q \leq 1$ and $p, q \in \mathbb{Q}\}$ as follows:*

$$(\langle p_1, q_1\rangle, \ldots, \langle p_n, q_n\rangle) \in S_f^\varepsilon \iff \forall i \leq n(q_i - p_i \leq 1/i) \textit{ and } \cap_{i=1}^n [p_i, q_i] \neq \emptyset \\ \textit{and } \forall i < n(|\Delta_f(p_{i+1}, q_{i+1}) - \Delta_f(p_i, q_i)| \geq \varepsilon).$$

Kechris and Woodin showed that for all $f \in C[0,1]$, $f \in \mathcal{D}$ if and only if $\forall \varepsilon \in \mathbb{Q}^+(S_f^\varepsilon$ is well-founded). That makes possible the following alternative rank definition:

Definition 2.8. *Let $f \in \mathcal{D}$. Define $|f|^* = \sup\{|S_f^\varepsilon| + 1 : \varepsilon \in \mathbb{Q}^+\}$.*

Lemma 2.2. *If $f \in \mathcal{D}$ is computable, then $|f|^*$ is constructive.*

Proof. Note that the tree S_f^ε would be computable if one did not have to verify that $|\Delta_f(p_{i+1}, q_{i+1}) - \Delta_f(p_i, q_i)| \geq \varepsilon$, a Π_1 statement. In fact such a strong statement is not needed, and to get around it we use a computable approximation. For any computable $g \in C[0,1]$, rational $p \in [0,1]$, and rational $\delta > 0$, the notation $[g(p)]_\delta$ refers to a standard δ-approximation of $g(p)$, which is a rational number z such that $|g(p) - z| < \delta$. (For specificity we could say $[g(p)]_\delta$ is the b component of the smallest $\langle n, p, r, b, \delta/2\rangle$ in the computable code for g.) Given a computable f, consider the following collection of trees $\tilde{S}_f^\varepsilon$, which are the same as the S_f^ε defined above, except for the use of a computable approximation:

$$(\langle p_1, q_1\rangle, \ldots, \langle p_n, q_n\rangle) \in \tilde{S}_f^\varepsilon \iff \forall i \leq n(q_i - p_i \leq 1/i) \text{ and } \cap_{i=1}^n [p_i, q_i] \neq \emptyset \\ \text{and } \forall i < n([|\Delta_f(p_{i+1}, q_{i+1}) - \Delta_f(p_i, q_i)|]_{\varepsilon/4} \geq \varepsilon).$$

The $\tilde{S}_f^\varepsilon$ are computable trees, uniformly in f and ε. Furthermore, for each ε, $S_f^{2\varepsilon} \subseteq \tilde{S}_f^\varepsilon \subseteq S_f^{\varepsilon/2}$, so $|S_f^{2\varepsilon}| \leq |\tilde{S}_f^\varepsilon| \leq |S_f^{\varepsilon/2}|$. Therefore, although $|f|^*$ is defined in terms of S_f^ε, it is also true that $|f|^* = \sup\{|\tilde{S}_f^\varepsilon| + 1 : \varepsilon \in \mathbb{Q}^+\}$. Since $\tilde{S}_f^\varepsilon$ are defined uniformly in ε, the tree

$$\tilde{S} = \{\langle\varepsilon\rangle^\frown\sigma : \varepsilon \in \mathbb{Q}^+, \sigma \in \tilde{S}_f^\varepsilon\}$$

is also computable, and $|f|^* = |\tilde{S}|$. Therefore $|f|^*$ is constructive. □

Theorem 2.3 ([KW86]). *Let $f \in \mathcal{D}$. Then if f is linear, $|f|_{KW} = 1$, and if f is not linear, $|f|^* = \omega|f|_{KW}$.*

Therefore, for each computable f, $|f|_{KW} \in \mathcal{O}$. Thus

$$D = \bigcup_{\alpha < \omega_1^{CK}} D_\alpha.$$

By the standard definition of differentiability, D is a Π_1^1 set. Mazurkiewicz [Maz36] gave a reduction from well-founded trees to differentiable functions. This reduction, reproduced in [KW86], easily effectivizes, and therefore also serves as a reduction from $\mathcal{O}$ to D. Thus we know that D is Π_1^1-complete. We will generate functions from well-founded trees using a method similar to that of Mazurkiewicz. By constructing the trees carefully we can obtain finer grained results.

3. Having rank at most α is $\Pi_{2\alpha+1}$

In this section, we show that "$|f|_{KW} < \alpha + 1$" is a $\Pi_{2\alpha+1}$ statement. This follows from a mostly straightforward translation of the definition of differentiability rank into the formal language. The only obstacle is that the original definition needs to be slightly optimized. In Section 3.1 we give an equivalent definition of differentiability rank which uses fewer quantifiers. In Section 3.2 we formalize the sentence "$|f|_{KW} \leq \alpha + 1$".

3.1. **An equivalent rank function.** In [KW86] the rank is defined using a "derivative operation" $P'_{f,\varepsilon}$ on sets P. To prove our result we use an almost identical operation $P^*_{f,\varepsilon}$ defined below. The only difference between this definition and the definition of $P'_{f,\varepsilon}$ is that $\geq$ is replaced with $>$. This is done in order to make the statement $[0,1]^*_{f,\varepsilon} = \emptyset$ a Σ_2 statement (instead of Σ_3), and this is necessary for the base case of Proposition 3.3.

Definition 3.1. *Given a closed set P, a function f and $\varepsilon > 0$,*

$$P^*_{f,\varepsilon} = \{x \in P : \forall \delta > 0 \exists p < q, r < s \in B(x,\delta) \cap [0,1] \\ \text{with } [p,q] \cap [r,s] \cap P \neq \emptyset \text{ and } |\Delta_f(p,q) - \Delta_f(r,s)| > \varepsilon\}$$

where all the quantifiers range over rational numbers.

It is easy to see that $P^*_{f,\varepsilon}$ is a closed subset of P, so it makes sense to define a rank function using it. We define a hierarchy of closed sets analogously to [KW86]:

Definition 3.2. ($\tilde{P}^\alpha_{f,\varepsilon}(I)$ hierarchy) *Fix a continuous function f, a rational $\varepsilon > 0$, and a closed set $I \subseteq [0,1]$. Define $\tilde{P}^0_{f,\varepsilon}(I) = I$. Then for each ordinal α, define $\tilde{P}^{\alpha+1}_{f,\varepsilon}(I) = (\tilde{P}^\alpha_{f,\varepsilon}(I))^*_{f,\varepsilon}$. If λ is a limit ordinal, define $\tilde{P}^\lambda_{f,\varepsilon}(I) = \cap_{\alpha<\lambda} \tilde{P}^\alpha_{f,\varepsilon}(I)$.*

In the special case $I = [0,1]$, we write $\tilde{P}^\alpha_{f,\varepsilon}$ instead of $\tilde{P}^\alpha_{f,\varepsilon}([0,1])$. Sometimes the function f may also be omitted from the notation if it is clear from context.

The rank of a differentiable function f is defined in [KW86] to be the smallest ordinal α such that for all ε, $P^\alpha_\varepsilon = \emptyset$. The next lemma shows our $\tilde{P}^\alpha_\varepsilon$ hierarchy is similar enough to preserve the notion.

Lemma 3.1. *For any differentiable $f \in C[0,1]$, $\varepsilon > 0$ and ordinal α,*

$$\tilde{P}^\alpha_\varepsilon \subseteq P^\alpha_{\varepsilon/2} \subseteq \tilde{P}^\alpha_{\varepsilon/4}.$$

Proof. The proof is by induction on α. When $\alpha = 0$ all these sets coincide. Next we observe that both $'$ and * have the property that if $P \subseteq Q$, then for any ε, $P'_\varepsilon \subseteq Q'_\varepsilon$ and $P^*_\varepsilon \subseteq Q^*_\varepsilon$. Also it is easy to observe that for all ε and all P, $P^*_\varepsilon \subseteq P'_{\varepsilon/2} \subseteq P^*_{\varepsilon/4}$. So when $\alpha = \beta + 1$, if we assume $\tilde{P}^\beta_\varepsilon \subseteq P^\beta_{\varepsilon/2} \subseteq \tilde{P}^\beta_{\varepsilon/4}$ we have

$$\tilde{P}^\alpha_\varepsilon = (\tilde{P}^\beta_\varepsilon)^*_\varepsilon \subseteq (P^\beta_{\varepsilon/2})^*_\varepsilon \subseteq (P^\beta_{\varepsilon/2})'_{\varepsilon/2} = P^\alpha_{\varepsilon/2}$$
$$P^\alpha_{\varepsilon/2} = (P^\beta_{\varepsilon/2})'_{\varepsilon/2} \subseteq (\tilde{P}^\beta_{\varepsilon/4})'_{\varepsilon/2} \subseteq (\tilde{P}^\beta_{\varepsilon/4})^*_{\varepsilon/4} = \tilde{P}^\alpha_{\varepsilon/4}$$

Finally, when λ is a limit, $\cap_{\alpha<\lambda} \tilde{P}^\alpha_\varepsilon \subseteq \cap_{\alpha<\lambda} P^\alpha_{\varepsilon/2} \subseteq \cap_{\alpha<\lambda} \tilde{P}^\alpha_{\varepsilon/4}$ follows because $\tilde{P}^\alpha_\varepsilon \subseteq P^\alpha_{\varepsilon/2} \subseteq \tilde{P}^\alpha_{\varepsilon/4}$ holds for all $\alpha < \lambda$. $\square$

From Lemma 3.1 it is clear that for all α,

$$\forall\varepsilon P^\alpha_\varepsilon = \emptyset \iff \forall\varepsilon \tilde{P}^\alpha_\varepsilon = \emptyset,$$

and thus the notion of rank defined using the P^α_ε hierarchy coincides with the notion of rank defined using the $\tilde{P}^\alpha_\varepsilon$ hierarchy.

3.2. **The formal statements "$|\mathbf{f}|_{\mathbf{KW}} \leq \alpha + \mathbf{1}$".** Before we can use the previous section's definition to formalize "$|f|_{KW} \leq \alpha + 1$", we need the following lemma. Briefly, the lemma holds because membership in $\tilde{P}^\alpha_\varepsilon(I)$ is a local property.

Lemma 3.2. *Fix f and ε. For any closed $I \subseteq [0,1]$, any closed interval $[i,j]$, and any α,*

$$[i,j] \cap \tilde{P}^\alpha_\varepsilon(I) = \bigcap_{d>0} \tilde{P}^\alpha_\varepsilon([i-d, j+d] \cap I).$$

Proof. On the one hand, suppose that $x \notin [i,j] \cap \tilde{P}^\alpha_\varepsilon(I)$. If $x \notin [i,j]$ then eventually $x \notin [i-d, j+d]$. So assume that $x \in [i,j]$. Then $x \notin \tilde{P}^\alpha_\varepsilon(I)$, so x could not be in $\tilde{P}^\alpha_\varepsilon([i-d,j+d] \cap I)$ for any d, since $\tilde{P}^\alpha_\varepsilon([i-d,j+d] \cap I) \subseteq \tilde{P}^\alpha_\varepsilon(I)$ for all α.

For the other direction we proceed by induction on α. The relationship certainly holds when $\alpha = 0$. Suppose $\alpha = \beta + 1$ and suppose that $x \in [i,j] \cap \tilde{P}^\alpha_\varepsilon(I)$. We wish to show that $x \in \tilde{P}^\alpha_\varepsilon([i-d, j+d] \cap I)$, so fix δ, and we will proceed to find our witnesses. Since $x \in \tilde{P}^\alpha_\varepsilon(I)$, let $p < q, r < s \in B(x, \min(\delta, d/2)) \cap I$ be such that $[p,q] \cap [r,s] \cap \tilde{P}^\beta_\varepsilon(I) \neq \emptyset$ and $|\Delta_f(p,q) - \Delta_f(r,s)| > \varepsilon$. Then because $x \in [i,j]$, we have these same $p,q,r,s \in B(x,\delta) \cap [i-d, j+d] \cap I$, and in fact, because p,q,r,s are within $d/2$ of $[i,j]$, we have $p,q,r,s \in [i-d/2, j+d/2]$. If we can show that $[p,q] \cap [r,s] \cap \tilde{P}^\beta_\varepsilon([i-d,j+d] \cap I) \neq \emptyset$ then we are done.

Let $z \in [p,q] \cap [r,s] \cap \tilde{P}^\beta_\varepsilon(I)$. By the induction hypothesis,

$$z \in \bigcap_{\zeta>0} \tilde{P}^\beta_\varepsilon([\max(p,r) - \zeta, \min(q,s) + \zeta] \cap I).$$

So in particular

$$z \in \tilde{P}^\beta_\varepsilon([\max(p,r) - d/2, \min(q,s) + d/2] \cap I) \subseteq \tilde{P}^\beta_\varepsilon([i-d, j+d] \cap I).$$

This completes the proof for the successor case.

Finally, if α is a limit ordinal, we have

$$\begin{aligned}
[i,j] \cap \tilde{P}^\alpha_\varepsilon(I) &= \bigcap_{\beta<\alpha} [i,j] \cap \tilde{P}^\beta_\varepsilon(I) \\
&= \bigcap_{\beta<\alpha} \bigcap_{d>0} \tilde{P}^\beta_\varepsilon([i-d, j+d] \cap I) \\
&= \bigcap_{d>0} \bigcap_{\beta<\alpha} \tilde{P}^\beta_\varepsilon([i-d, j+d] \cap I) \\
&= \bigcap_{d>0} \tilde{P}^\alpha_\varepsilon([i-d, j+d] \cap I).
\end{aligned}$$

□

The definition of the rank of a function f uses transfinite recursion in order to calculate $P^\alpha_{f,\varepsilon}$ for each α while holding ε fixed. Thus, knowing the expressive complexity of "$|f|_{KW} \leq 1$" does not give us a foothold into the expressive complexity of "$|f|_{KW} \leq 2$", because "$|f|_{KW} \leq \alpha$" does not appear as a sub-expression of

"$|f|_{KW} \leq \alpha+1$". The sub-expression which does persist, and on which it is almost appropriate to transfinitely recurse, is "$[i,j] \cap \tilde{P}^\alpha = \emptyset$", where $[i,j]$ is some arbitrary interval. Lemma 3.2 lets us express this intersection in statements of the form "$\tilde{P}^\alpha([i,j]) = \emptyset$", and so this last expression is a useful core concept. Its expressive complexity is $\Sigma_{2\alpha}$, as seen in the next proposition.

Proposition 3.3. *Let $\alpha > 0$ be a constructive ordinal, $\varepsilon, i, j \in \mathbb{Q}$ with $\varepsilon > 0$ and $0 \leq i < j \leq 1$. The set of e such that $\tilde{P}^\alpha_{f_e,\varepsilon}([i,j]) = \emptyset$ is $\Sigma_{2\alpha}$, uniformly in α, ε, i and j.*

Proof. We carry along an arbitrary index e and oscillation sensitivity ε, so to reduce clutter we write f instead of f_e, and $\tilde{P}^\alpha$ instead of $\tilde{P}^\alpha_{f,\varepsilon}$.

In general, when $\alpha = \beta + 1$,

$$\begin{aligned}\tilde{P}^\alpha([i,j]) = [i,j] \setminus \bigcup \{ I : \forall p,q,r,s \in I \\ \Big([p,q] \cap [r,s] \cap \tilde{P}^\beta([i,j]) = \emptyset \vee |\Delta_f(p,q) - \Delta_f(r,s)| \leq \varepsilon \Big) \}\end{aligned}$$

where I ranges over intervals open in $[i,j]$. Since the I are closed under taking subsets, it suffices to let I range over intervals open in $[i,j]$ with rational endpoints. So $\tilde{P}^\alpha([i,j]) = \emptyset$ if and only if these I cover $[i,j]$. If the I do cover, then by compactness there is a rational δ such that for all $x \in [i,j]$, $B(x,\delta) \subseteq I$ for some I. Thus there is a δ such that for any open interval U with rational endpoints where $\operatorname{diam}(U) < \delta$, $U \subseteq I$ for some I. On the other hand, if the I do not cover, then there cannot be any such δ. Thus if $\alpha = \beta + 1$,

$$\begin{aligned}\tilde{P}^\alpha([i,j]) = \emptyset \iff \exists \delta > 0 \forall c \in [i,j] \forall p,q,r,s \in B(c,\delta) \cap [i,j] \\ \Big([p,q] \cap [r,s] \cap \tilde{P}^\beta([i,j]) = \emptyset \vee |\Delta_f(p,q) - \Delta_f(r,s)| \leq \varepsilon \Big)\end{aligned}$$

where all quantifiers range over the rationals.

When $\beta = 0$, $[p,q] \cap [r,s] \cap \tilde{P}^\beta([i,j]) = \emptyset \Leftrightarrow [p,q] \cap [r,s] \cap [i,j] = \emptyset$, so the above statement is Σ_2 uniformly in e, ε, i, and j.

When $\beta > 0$, we have

$$\begin{aligned}&[p,q] \cap [r,s] \cap \tilde{P}^\beta([i,j]) = \emptyset \\ &\qquad \iff \exists \zeta \tilde{P}^\beta([\max(p,r) - \zeta, \min(q,s) + \zeta] \cap [i,j]) = \emptyset.\end{aligned}$$

which follows from Lemma 3.2 and compactness. Thus with the assumption that $\tilde{P}^\beta([c,d]) = \emptyset$ is $\Sigma_{2\beta}$ uniformly in all variables, then $\tilde{P}^{\beta+1}([i,j]) = \emptyset$ is $\Sigma_{2\beta+2}$, uniformly in all variables.

Finally, suppose that α is a limit, given as a uniform supremum $\alpha = \sup_n \beta_n$. Then by compactness and the definition of $\tilde{P}^\alpha$ for α a limit,

$$P^\alpha([i,j]) = \emptyset \iff \exists n \tilde{P}^{\beta_n}([i,j]) = \emptyset.$$

So assuming that $\tilde{P}^{\beta_n}([i,j]) = \emptyset$ is uniformly $\Sigma_{2\beta_n}$ in all variables including n, we see that $\tilde{P}^\alpha([i,j]) = \emptyset$ is uniformly Σ_α, which is the same as $\Sigma_{2\alpha}$ since α is a limit. □

Proposition 3.4. *For any constructive $\alpha > 0$, $D_{\alpha+1}$ is $\Pi_{2\alpha+1}$, uniformly in α.*

Proof. We have

$$e \in D_{\alpha+1} \iff f_e \in C[0,1] \wedge \forall \varepsilon [\tilde{P}^{\alpha}_{f_e,\varepsilon} = \emptyset]$$

where ε ranges over positive rationals. Recall that "$f_e \in C[0,1]$" is Π_2, and by Proposition 3.3, $\tilde{P}^{\alpha}_{f_e,\varepsilon} = \emptyset$ is $\Sigma_{2\alpha}$. Thus the right hand side is a $\Pi_{2\alpha+1}$ statement, uniformly in α and e. □

4. Having rank at most α is $\Pi_{2\alpha+1}$-complete

In this section, we provide a many-one reduction in the other direction, from $\overline{\emptyset_{(2\alpha+1)}}$ to $D_{\alpha+1}$. The most significant step of the reduction is found in Section 4.3. To set up this step, we happen to need only a certain class of $C[0,1]$ functions which can be structurally represented by well-founded trees according to a recipe reminiscent of Mazurkiewicz's original reduction. This allows us to construct a function of the right rank through an intermediate step of constructing a tree with the right structure.

In Section 4.1 we construct special $C[0,1]$ functions which reflect the structure of well-founded trees on $\mathbb{N}^{<\mathbb{N}}$. In Section 4.2, we define a rank on well-founded trees which agrees with the differentiability rank of the functions that the trees generate, when a fixed arbitrary value for ε is used. In Section 4.3 we give a reduction from canonical complete sets to trees of an appropriate rank, obtaining a result one jump short of the final result. Section 4.4 combines the results of the previous sections with the additional ingredient of varying ε to obtain the final result.

4.1. **Making differentiable functions out of well-founded trees.** The idea of this section is to set up countably many closed disjoint intervals in $[0,1]$, put the intervals in bijective correspondence with $\mathbb{N}^{<\mathbb{N}}$, and then given a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, define f_T as a sum of continuously differentiable bumps supported on each of the intervals which correspond to $\sigma \in T$. These functions are structurally similar to the ones described in Section 2.4. If $S = \{\rho : \sigma^\frown\rho \in T\}$ then a shrunken version of f_S can be found in f_T. Furthermore, if $\tau \supset \sigma$, then the bump corresponding to τ is in the shadow of the bump corresponding to σ. The intervals are arranged so that the resulting f_T has a differentiability rank which can be computed from T in a way that is described in the next section.

In the following definition, the choices of the constants $\frac{1}{2}$ and $\frac{1}{4}$ and the bounds on p and p' are arbitrary, but consistent with each other. The requirement $b_n - a_n < (a_n - \frac{1}{4})^2$ is what keeps f_T everywhere differentiable.

Definition 4.1. *Let $p : [0,1] \to \mathbb{R}$ be a computable function satisfying*

(1) *p is continuously differentiable*
(2) $p(\frac{1}{2}) = \frac{1}{2}$
(3) $p(0) = p(1) = p'(0) = p'(1) = 0$
(4) $||p|| < 1$ *and* $||p'|| < 2$

Let $\{[a_n, b_n]\}_{n\in\mathbb{N}}$ be any computable sequence of intervals with rational endpoints satisfying

(1) *Each interval is contained in $(\frac{1}{4}, \frac{1}{2})$*
(2) *$b_{n+1} < a_n < b_n$ for each n.*
(3) $\lim_{n\to\infty} a_n = \frac{1}{4}$
(4) *$b_n - a_n < (a_n - \frac{1}{4})^2$ for each n*

Then for any well-founded tree $T \in \mathbb{N}^{<\mathbb{N}}$, define f_T as follows.

(1) *If T is empty, $f_T \equiv 0$.*
(2) *Otherwise, $f_T = p[\frac{1}{2}, 1] + \sum_{n=0}^{\infty} f_{T_n}[a_n, b_n]$*

Recall that $f[a, b]$ denotes a copy of f proportionally resized to have domain $[a, b]$, and that T_n denotes $\{\sigma : \langle n\rangle^\frown\sigma \in T\}$, the nth subtree of T. Now we verify that the above definition produces well-defined computable differentiable functions.

Proposition 4.1. *For any well-founded computable tree $T \in \mathbb{N}^{<\mathbb{N}}$:*

(1) *f_T is uniformly computable in T*
(2) *f_T is differentiable*
(3) *$f_T(0) = f_T(1) = f'_T(0) = f'_T(1) = 0$*
(4) *$||f_T|| < 1$ and $||f'_T|| < 2$*

Proof. Proceeding by induction on the rank of the tree, in the base case all four properties are satisfied. Assume they hold for all trees of rank less than $|T|$. Then the sequence f_{T_n} is uniformly computable with each $||f_{T_n}|| < 1$. Then on any interval whose closure does not contain $\frac{1}{4}$, f_T is equal to a uniformly determined finite sum of computable functions, and is thus computable. And for ε sufficiently small, we claim that $|f_T\left((\frac{1}{4} - \varepsilon, \frac{1}{4} + \varepsilon)\right)| < \varepsilon^2$. This follows because $||f_{T_n}|| < 1$ by induction and because the intervals $[a_n, b_n]$ are disjoint, so for any x in such an interval we have the bound

$$|f_T(x)| = f_{T_n}[a_n, b_n](x) \leq ||f_{T_n}||(b_n - a_n) \leq 1 \cdot (a_n - \frac{1}{4})^2 \leq (x - \frac{1}{4})^2 < \varepsilon^2.$$

Therefore f_T is uniformly computable in T. Similarly, assuming f_{T_n} are each differentiable with $f_{T_n}(0) = f_{T_n}(1) = f'_{T_n}(0) = f'_{T_n}(1) = 0$, then each $f_{T_n}[a_n, b_n]$ is differentiable. Then f_T is certainly differentiable at any point $x \neq \frac{1}{4}$, since on some neighborhood of that point f_T is equal to a finite sum of differentiable functions. On the other hand, in the vicinity of $\frac{1}{4}$, f_T satisfies $|f_T(x)| \leq (x - \frac{1}{4})^2$, so f_T is differentiable at $\frac{1}{4}$ as well. Because $f_T \upharpoonright [0, \frac{1}{4}] \equiv 0$ and $p(1) = p'(1) = 0$, we have $f_T(0) = f_T(1) = f'_T(0) = f'_T(1) = 0$. Finally, $||f_T|| < 1$ and $||f'_T|| < 2$ by induction, because $||p|| < 1$,$||p'|| < 2$, and $||f_{T_n}|| < 1$,$||f'_{T_n}|| < 2$, for each n, and the shrunken copies $p[\frac{1}{2}, 1]$ and $f_{T_n}[a_n, b_n]$ have disjoint support. □

We close this section with some comments about why this f_T is defined as it is, using the concepts from Section 2.4. Note that for every nonempty S,

$$\Delta_{f_S}(0, \frac{3}{4}) = \frac{1}{3} \text{ and } \Delta_{f_S}(0, \frac{1}{2}) = 0.$$

Now for each n, $f_{T_n}[a_n, b_n]$ is a proportionally shrunken copy of f_{T_n}, so unless T_n is empty, $f_{T_n}[a_n, b_n]$ contributes a bump and its pair of secants with slopes 0 and $\frac{1}{3}$. Thus a tree with infinitely many children of the root has infinitely many pairs of these disagreeing secants. If we construct T_n so that f_{T_n} has a large rank, the nth disagreeing pair of secants will be visible for many iterations of the rank-ascertaining process for f_T, because $P^\alpha_{f_T} \cap [a_n, \frac{a_n+b_n}{2}]$ will be nonempty for many iterations. If we construct T so that f_{T_n} has large rank for infinitely many n, these disagreeing pairs of secants can make a contribution to raising the Kechris-Woodin rank of f_T. This can happen in two ways: if $|f_{T_n}|_{KW} = \alpha + 1$ for infinitely many n, then $|f_T| \geq \alpha + 2$. And if the ranks of the f_{T_n} are unbounded below a limit ordinal λ, then $|f_T|_{KW} = \lambda + 1$.

4.2. **A rank on well-founded trees which agrees with the differentiability rank of the corresponding functions.** We will now show that when a function is generated from a tree in the way described above, its Kechris-Woodin rank can be read right off the tree. Furthermore, we will see that this function's rank can already be witnessed at a fixed oscillation sensitivity $\varepsilon = \frac{1}{4}$. That is, the rank of f_T is always a successor, and when $|f_T|_{KW} = \alpha + 1$, then $\tilde{P}^{\alpha}_{f,\frac{1}{4}} \neq \emptyset$. Here is a rank on trees which corresponds to the differentiability rank of the functions they generate.

Definition 4.2. *For a well-founded tree $T \in \mathbb{N}^{<\mathbb{N}}$, define the limsup rank of the tree by*

$$|T|_{ls} = \max\left(\sup_n |T_n|_{ls}, (\limsup_n |T_n|_{ls}) + 1\right),$$

if T is nonempty, and $|T|_{ls} = 0$ if T is empty.

Note that reordering the subtrees does not change the limsup rank of the tree. A node can have a rank higher than all its children in one of two situations: either there is no child of maximal rank, or there are infinitely many maximal rank children. In the next proposition, we will see that this mechanism corresponds exactly to the mechanism for constructing functions of increasing differentiability rank.

The following two straightforward lemmas which we will use later are woven into the proof of Fact 3.5 in [KW86]. For the purposes of exposition, we state and prove them here.

Lemma 4.2. *If $U \subseteq [0,1]$ is open and $f \restriction U = g \restriction U$, then for all α and ε, $P^{\alpha}_{f,\varepsilon} \cap U = P^{\alpha}_{g,\varepsilon} \cap U$.*

Proof. By induction on α. The base and limit cases are trivial. Suppose that $P^{\alpha}_{f,\varepsilon} \cap U = P^{\alpha}_{g,\varepsilon} \cap U$. Fix $x \in U$ and let λ be small enough that $B(X, \lambda) \subseteq U$. Then $x \in P^{\alpha+1}_{f,\varepsilon}$ if and only if for all $\delta < \lambda$ there are $p, q, r, s \in B(x, \delta)$ such that $|\Delta_f(p,q) - \Delta_f(r,s)| \geq \varepsilon$ and $[p,q] \cap [r,s] \cap P^{\alpha}_{f,\varepsilon} \neq \emptyset$. Since $p, q, r, s \in B(x,\delta) \subseteq B(x, \lambda) \subseteq U$, we have $[p,q] \cap [r,s] \cap P^{\alpha}_{f,\varepsilon} \neq \emptyset$ if and only if $[p,q] \cap [r,s] \cap P^{\alpha}_{g,\varepsilon} \neq \emptyset$. Thus $x \in P^{\alpha+1}_{f,\varepsilon}$ if and only if $x \in P^{\alpha+1}_{g,\varepsilon}$. □

Recall that for any function $f \in C[0,1]$, we write $f[a,b]$ to denote a proportionally shrunken verson of f. By definition, $f[a,b]$ is the function which is identically 0 outside of $[a,b]$, and for $x \in [a,b]$, $f[a,b](x) = (b-a)f(\frac{x-a}{b-a})$. Similarly, for any real number $c \in [0,1]$ and any interval $[a,b]$, let $c[a,b] = a + c(b-a)$. The point is that c is to f as $c[a,b]$ is to $f[a,b]$.

Lemma 4.3. *Let $f \in C[0,1]$ be a differentiable function satisfying $f(0) = f(1) = f'(0) = f'(1) = 0$. Let $[a,b] \subseteq [0,1]$ be an interval with rational endpoints. Then $|f|_{KW} = |f[a,b]|_{KW}$. Furthermore, for any ordinal α and for all $x \in [0,1]$,*

(1) $x \in P^{\alpha}_{f,\varepsilon} \implies x[a,b] \in P^{\alpha}_{f[a,b],\varepsilon}$
(2) $x[a,b] \in P^{\alpha}_{f[a,b],\varepsilon} \implies x \in P^{\alpha}_{f,\varepsilon/2}$

Proof. Proceeding by induction, it is clear that the both items holds when $\alpha = 0$. The limit case is also trivial.

Assume the first item holds for some α. If $x \in P^{\alpha+1}_{f,\varepsilon}$, then the collection of all the tuples p, q, r, s which witness this can be mapped to a collection of tuples $p[a,b], q[a,b], r[a,b], s[a,b]$ which witness $x[a,b] \in P^{\alpha+1}_{f[a,b],\varepsilon}$. That proves the first item.

On the other hand, suppose the second item holds for some α. If $x[a,b] \in P^{\alpha+1}_{f[a,b],\varepsilon}$ and $x \in (0,1)$ (i.e. x is not an endpoint), then as above corresponding witnesses can always be chosen for sufficiently small neighborhoods of x, so $x \in P^{\alpha+1}_{f,\varepsilon} \subseteq P^{\alpha+1}_{f,\varepsilon/2}$. Last we consider the endpoint case: suppose $a \in P^{\alpha+1}_{f[a,b],\varepsilon}$ (and the case $b \in P^{\alpha+1}_{f[a,b],\varepsilon}$ is of course just the same). Because $a = 0[a,b]$ and $f'(0) = 0$, let λ be small enough that for all distinct $p, q \in B(a,\lambda)$ with $p \le a \le q$, $|\Delta_{f[a,b]}(p,q)| < \varepsilon/4$. Then for each $\delta > 0$, there are $p,q,r,s \in B(a, \min(\lambda,(b-a)\delta))$ such that $|\Delta_{f[a,b]}(p,q) - \Delta_{f[a,b]}(r,s)| \ge \varepsilon$ and $[p,q] \cap [r,s] \cap P^{\alpha}_{f[a,b],\varepsilon} \neq \emptyset$. Then without loss of generality, $|\Delta_{f[a,b]}(p,q)| \ge \varepsilon/2$, so $a < p < q$. If also $a < r < s$, then we are done since the corresponding $\frac{p-a}{b-a}$, etc. can be used as the witness for δ. It is impossible that $r < s < a < p < q$ because $[p,q] \cap [r,s] \neq \emptyset$. In the last case, if $r < a < s$, this implies that $|\Delta_{f[a,b]}(r,s)| < \varepsilon/4$, so $|\Delta_{f[a,b]}(p,q)| \ge 3\varepsilon/4$. But then also $|\Delta_{f[a,b]}(a,s)| < \varepsilon/4$, and thus $|\Delta_{f[a,b]}(p,q) - \Delta_{f[a,b]}(a,s)| \ge \varepsilon/2$. Also $[p,q] \cap [a,s] = [p,q] \cap [r,s]$, and there is some $y \in [p,q] \cap [a,s] \cap P^{\alpha}_{f[a,b],\varepsilon}$, and by induction $\frac{y-a}{b-a} \in P^{\alpha}_{f,\varepsilon/2}$. Therefore $\frac{p-a}{b-a}, \frac{q-a}{b-a}, 0, \frac{s-a}{b-a}$ will do, and thus $x \in P^{\alpha+1}_{f,\varepsilon/2}$.

Finally, note that by the previous lemma, $P^{\alpha}_{f[a,b],\varepsilon} \cap ([0,1] \setminus [a,b]) = \emptyset$ for any $\alpha > 0$. Therefore, $|f|_{KW} = |f[a,b]|_{KW}$. □

The next proposition shows that for any well-founded tree T, the differentiable function f_T defined in the previous section has rank $|f_T|_{KW} = |T|_{ls}$, and that the rank of f_T is witnessed at oscillation sensitivity $\varepsilon = \frac{1}{4}$.

Proposition 4.4. *For any nonempty well-founded tree $T \in \mathbb{N}^{<\mathbb{N}}$,*

(1) *$|T|_{ls}$ is a successor,*
(2) *The function f_T is differentiable with $|f_T|_{KW} = |T|_{ls}$, and*
(3) *Letting $|T|_{ls} = \alpha + 1$, we have $P^{\alpha}_{f,\frac{1}{4}} \neq \emptyset$.*

Proof. The proof is by induction on the usual rank of the tree. If T is just a root (smallest option for the rank of the the tree since the statement is for nonempty trees only) then f_T is just $p[\frac{1}{2}, 1]$, so it is continuously differentiable with $|f_T|_{KW} = 1$. For each n, $T_n = \emptyset$ so $|T_n|_{ls} = 0$ so $\sup_n |T_n|_{ls} = \limsup_n |T_n|_{ls} = 0$, so $|T|_{ls} = 1$.

If T is more than a root, assume the lemma holds for each of the subtrees T_n. First we show that $|f_T|_{KW} \ge |T|_{ls}$. Fix n and let $|T_n|_{ls} = \alpha + 1$. Then by the inductive hypothesis $|f_{T_n}|_{KW} = \alpha + 1$ and $P^{\alpha}_{f_{T_n},\frac{1}{4}} \neq \emptyset$. By Lemma 4.3, $x \in P^{\alpha}_{f_{T_n},\frac{1}{4}} \implies x[a,b] \in P^{\alpha}_{f_{T_n}[a_n,b_n],\frac{1}{4}}$, so $P^{\alpha}_{f_{T_n}[a_n,b_n],\frac{1}{4}} \neq \emptyset$. Because the $[a_n, b_n]$ are closed and disjoint from each other and from $[\frac{1}{2}, 1]$, there is an $\varepsilon > 0$ such that $f_T \upharpoonright (a_n - \varepsilon, b_n + \varepsilon) = f_{T_n}[a_n, b_n] \upharpoonright (a_n - \varepsilon, b_n + \varepsilon)$, and therefore using Lemma 4.2, $P^{\alpha}_{f_T,\frac{1}{4}} \cap (a_n - \varepsilon, b_n + \varepsilon) = P^{\alpha}_{f_{T_n}[a_n,b_n],\frac{1}{4}} \cap (a_n - \varepsilon, b_n + \varepsilon) \neq \emptyset$. Therefore $P^{\alpha}_{f_T,\frac{1}{4}} \neq \emptyset$ and thus $|f_T|_{KW} \ge \alpha + 1$. So $|f_T|_{KW} \ge \sup_n |T_n|_{ls}$.

Now let us show that $|f_T|_{KW} \ge (\limsup_n |T_n|_{ls}) + 1$. Let $\alpha = \limsup_n |T_n|_{ls}$. We will show that $\frac{1}{4} \in P^{\alpha}_{f_T,\frac{1}{4}}$. First we show that for any $\beta < \alpha$, $\frac{1}{4} \in P^{\beta}_{f_T,\frac{1}{4}}$. There are infinitely many n such that $|T_n|_{ls} > \beta$, so $P^{\beta}_{f_{T_n},\frac{1}{4}} \neq \emptyset$ for infinitely many n by the inductive hypothesis, so $P^{\beta}_{f_{T_n}[a_n,b_n],\frac{1}{4}} \neq \emptyset$ for infinitely many n by Lemma 4.3. By Lemma 4.2, $P^{\beta}_{f_{T_n}[a_n,b_n],\frac{1}{4}} \subseteq P^{\beta}_{f_T,\frac{1}{4}}$. Because $\lim_{n\to\infty} a_n = \frac{1}{4}$, and infinitely many $[a_n, b_n]$ contain an element of $P^{\beta}_{f_T,\frac{1}{4}}$, $\frac{1}{4}$ is a limit point of $P^{\beta}_{f_T,\frac{1}{4}}$. Because this set is closed, $\frac{1}{4}$ must be in it as well. Thus $\frac{1}{4} \in P^{\beta}_{f_T,\frac{1}{4}}$ for all $\beta < \alpha$. If α is as limit, this implies $\frac{1}{4} \in P^{\alpha}_{f_T,\frac{1}{4}}$, so $|f_T|_{KW} > \alpha$ if α is a limit. Now suppose α is a successor. Let $\alpha = \beta + 1$. Let U be a neighborhood of $\frac{1}{4}$, and let n be chosen such that $[a_n, b_n] \subseteq U$ and $P^{\beta}_{f_{T_n}[a_n,b_n],\frac{1}{4}} \neq \emptyset$. Then $\Delta_{f_T}(a_n, \frac{3}{4}[a_n, b_n]) = \frac{1}{3}$ and $\Delta_{f_T}(a_n, \frac{1}{2}[a_n, b_n]) = 0$, and $[a_n, \frac{3}{4}[a_n, b_n]] \cap [a_n, \frac{1}{2}[a_n, b_n]] \cap P^{\beta}_{f_T,\frac{1}{4}} \neq \emptyset$. Therefore $\frac{1}{4} \in P^{\beta+1}_{f_T,\frac{1}{4}}$, so again $|f_T|_{KW} > \alpha$. This completes the claim that $|f_T|_{KW} \geq |T|_{ls}$.

Now let us show that $|f_T|_{KW} \leq |T|_{ls}$. First, let $\alpha = \sup_n |T_n|_{ls}$. Note that $\alpha > 0$ because the case of T being only a root was already considered separately. For each n, $|T_n|_{ls} \leq \alpha$, so by induction $|f_{T_n}|_{KW} = |f_{T_n}[a_n, b_n]|_{KW} \leq \alpha$. So for each n and ε we have $P^{\alpha}_{f_{T_n}[a_n,b_n],\varepsilon} = \emptyset$ and also $P^{\alpha}_{p[\frac{1}{2},1],\varepsilon} = \emptyset$. Cover $[0,1] \setminus \{\frac{1}{4}\}$ with open intervals U such that each U intersects at most one of the $[a_n, b_n]$ or $[\frac{1}{2}, 1]$. Then for each such interval and each ε, $P^{\alpha}_{f_T,\varepsilon} \cap U = P^{\alpha}_{f_{T_n}[a_n,b_n],\varepsilon} \cap U = \emptyset$, or $P^{\alpha}_{f_T,\varepsilon} \cap U = P^{\alpha}_{p[\frac{1}{2},1],\varepsilon} \cap U = \emptyset$, respectively. Therefore, for all ε, $P^{\alpha}_{f_T,\varepsilon} \subseteq \{\frac{1}{4}\}$. If $\limsup_n |T_n|_{ls} = \sup_n |T_n|_{ls}$, then $|T|_{ls} = \alpha + 1$, so this is enough: $P^{\alpha+1}_{f_T,\varepsilon} = \emptyset$ for all ε.

On the other hand, suppose $\limsup_n |T_n|_{ls} < \sup_n |T_n|_{ls}$. Then $\alpha = |T|_{ls} = \sup_n |T_n|_{ls}$ is a successor, because the induction hypothesis guarantees $|T_n|_{ls}$ is always a successor, and therefore if the sup were a limit, it would be equal to the limsup. Let $\alpha = \beta + 1$. Then eventually $|T_n|_{ls} \leq \beta$. Let V be an open neighborhood of $\frac{1}{4}$ such that $[a_n, b_n] \cap V \neq \emptyset$ implies $|T_n|_{ls} \leq \beta$. Covering $V \setminus \{\frac{1}{4}\}$ with open intervals U as before, we find $P^{\beta}_{f_{T_n}[a_n,b_n],\varepsilon} \cap U = \emptyset$ for each such $U \subseteq V$ and each ε, so $P^{\beta}_{f_T,\varepsilon} \cap V \subseteq \{\frac{1}{4}\}$, so $P^{\beta+1}_{f_T,\varepsilon} \cap V = \emptyset$. Therefore $P^{\beta+1}_{f_T,\varepsilon} = \emptyset$. Thus $|f_T|_{KW} \leq \sup_n |T_n|_{ls}$. □

4.3. **Recognizing trees of limsup rank α is $\Sigma_{2\alpha}$-hard.** This section contains the core of the reduction. The goal of the section is to establish a reduction from $\Sigma_{2\alpha}$-complete sets to trees of rank $\leq \alpha$. This is one quantifier too few, but the result is also too strong – the functions produced from the resulting trees all reveal their rank at a uniform oscillation sensitivity $\varepsilon = \frac{1}{4}$. It will be a simple matter later to encode another quantifier by producing functions whose ranks are witnessed at non-uniform oscillation sensitivity. (Indeed, this is exactly the approach suggested by the definition of the rank.)

The purpose of the next two lemmas is to specify exactly how to strip two quantifiers off most Π_α facts in a particularly nice way, a way which will be useful for the main argument which is coming up in Lemma 4.7. The lemmas are surely known, but proofs are provided for completeness.

The first lemma takes an arbitrary $\Pi_{\alpha+2}$ fact and rewrites it in a nice form, with unique witnesses and stable evidence. In the process, two computable reduction functions g_0 and g_s are defined which will be used in Lemma 4.7.

Lemma 4.5. *For any $\Pi_{\alpha+2}$ predicate $P(x)$, there is a Π_α predicate $R(x, z, y)$ such that*

1. $P(x) \iff \forall z \exists y R(x, z, y)$
2. $R(x, z, y_1) \wedge R(x, z, y_2) \implies y_1 = y_2$ *(R has unique witnesses)*
3. *For* $z_1 < z_2$, $\neg\exists y R(x, z_1, y) \implies \neg\exists y R(x, z_2, y)$ *(R has stable evidence)*
4. $R(x, z, y) \implies z < y$

Proof. We may as well assume that $P(x)$ is "$x \notin \emptyset_{(\alpha+2)}$". For the case $\alpha = 0$, we define R using a computable, total $\{0,1\}$-valued function g_0, and set $R(x,z,y) \iff g_0(x,z,y) = 1$.

Let e be a Π_2 index for $\overline{\emptyset''}$, i.e. ϕ_e is total and $x \notin \emptyset'' \iff \forall v \exists w[\phi_e(x,v,w) = 1]$. Define

$$g_0(x,z,y) = \begin{cases} 1 & \text{if } y > z \text{ and for all } v < z \text{ there is } w < y \text{ such that} \\ & \phi_e(x,v,w) = 1 \text{ and } y \text{ is least such that this is true} \\ 0 & \text{otherwise.} \end{cases}$$

One may check that four conditions on R are satisfied.

For the case $\alpha > 0$, we define R using a computable, total function g_s and set $R(x,z,y) \iff g_s(x,z,y) \notin \emptyset_{(\alpha)}$. The construction that defines g_s uses movable markers to build Π_α sets with at most one element. At any moment there is one particular element being held which is linked to a potential least-witness, and this element will be held for as long as that witness seems viable.

Let e be a universal Π_3 index, i.e. ϕ_e^X is total for all X and

$$x \notin X''' \iff \forall u \exists v \forall w[\phi_e^X(x,u,v,w) = 1].$$

The intended oracle X is an inverse jump of $\emptyset_{(\alpha)}$, so that $X' = \emptyset_{(\alpha)}$ and $X''' = \emptyset_{(\alpha+2)}$. But the claims of the lemma also hold for an arbitrary X when we let $P(x)$ be $x \notin X'''$ and $R(x,z,y)$ be $g_s(x,z,y) \notin X'$.

Define $W^X_{g(x,z)}$ in stages according to the following dynamic process. At stage $s = 0$, let $W^X_{g(x,z),0} = \{n : n \leq z\}$, and let $t_1 = 0$. For each $s > 0$, let y^0_s and y^1_s be respectively the smallest and second smallest elements of $\overline{W^X_{g(x,z),s-1}}$. Check whether $(\forall u < z)(\exists v < t_s)(\forall w < s)[\phi_e^X(x,u,v,w) = 1]$. If this is so, put y^1_s into $W^X_{g(x,z),s}$, and set $t_{s+1} = t_s$. If this is not so, put y^0_s into $W^X_{g(x,z),s}$, and set $t_{s+1} = t_s + 1$.

Then define

$$W^X_{g_s(x,z,y)} = \begin{cases} \mathbb{N} & \text{if } y \in W^X_{g(x,z)} \\ \emptyset & \text{otherwise.} \end{cases}$$

This has the effect that $g_s(x,z,y) \in X' \leftrightarrow y \in W^X_{g(x,z)}$.

Now let us verify the claims of the lemma, in the more general case where $P(x)$ is $x \notin X'''$ and $R(x,z,y)$ is $g_s(x,z,y) \notin X'$.

First we address the second claim, that R has unique witnesses. For a given x, z, X, let us verify that there is at most one y such that $g_s(x,z,y) \notin X'$. Suppose y^0_s does not stabilize in the construction above. Then $\overline{W^X_{g(x,z)}}$ does not have a smallest element, so it is empty, so $W^X_{g(x,z)} = \mathbb{N}$. On the other hand if y^0_s stabilizes, then let s_0 be such that for all $s > s_0$, $y^0_{s_0} = y^0_s$. Then for all $s > s_0$, it must be that y^1_s is put into $W^X_{g(x,z),s}$, so $\overline{W^X_{g(x,z)}} = \{y^0_{s_0}\}$. Thus in either case, $W^X_{g_s(x,z,y)} = \mathbb{N}$ for all but at most one y, so $g_s(x,z,y) \in X'$ for all but at most one y.

For the first claim, suppose that $x \notin X'''$. This is true if and only if $\forall u \exists v \forall w[\phi_e^X(x,u,v,w) = 1]$ In that case, for all z, in the construction of $W^X_{g(x,z)}$, we see that t_s stabilizes, because there is a t for which $(\forall u < z)(\exists v < t)(\forall w)[\phi_e^X(x,u,v,w) = 1]$. And conversely, if t_s stabilizes for each z, then $x \notin X'''$. We have $\lim_s t_s$ exists exactly when $\lim_s y^0_s$ exists, since they always change together. And $\lim y^0_s = y$ exists exactly when

$\overline{W^X_{g(x,z)}} = \{y\}$, which is equivalent to saying $g_s(x,z,y) \notin X'$. Thus $x \notin X'''$ if and only if $g_s(x,z,y) \notin X'$.

For the third claim, note that if $z_1 < z_2$ then $\lim_s t_s(z_1) \leq \lim_s t_s(z_2)$ where $t_s(z)$ refers to the t_s-values associated to the construciton of $W^X_{g(x,z)}$. Thus, if $W^X_{g(x,z_1)} = \emptyset$, then $W^X_{g(x,z_2)} = \emptyset$ as well.

Finally, for the last claim, if $y \in \overline{W^X_{g(x,z)}}$ then $y > z$ because $\{n : n \leq z\} \subseteq W^X_{g(x,z)}$ from the outset.

□

The next lemma explicitly splits up the queries to a $\emptyset^{(\lambda)}$ oracle that occur during the evaluation of a Π_λ question. The goal is to isolate the parts of the computation that can be done using a weaker oracle. In the proof we define a function g_l which will be used in Lemma 4.7.

Lemma 4.6. *Let λ be a limit ordinal, given as a uniform supremum $\lambda = \sup_n \beta_n$. For any Π_λ predicate $P(x)$ there is a sequence of predicates R_n such that*

$$P(x) \iff \bigwedge_n R_n(x)$$

where R_n is $\Pi_{2\beta_n}$ for each n. Furthermore, the R_n are uniformly computable from P and λ.

Proof. We may assume that each β_n is a successor ordinal, and that $P(x)$ is "$x \notin \emptyset_{(\lambda)}$". Now we define R_n by specifying a computable function g_l below and letting $R_n(x) \iff g_l(x,\lambda,n) \notin \emptyset_{(2\beta_n)}$.

Uniformly in any pair of constructive ordinals $\alpha < \beta$, there is a reduction from $\emptyset_{(\beta)}$ to $\emptyset_{(\alpha)}$. (See for example [AK00, Lemma 5.1].) And any standard encoding will have the property that $\langle z, n\rangle \geq n$. Therefore, $\emptyset^{(\lambda)} \restriction n$ is uniformly computable from λ, n and $\emptyset_{(\beta_n)}$, in the sense that there is a partial recursive function $\sigma(\lambda, n, X)$ which halts and returns $\emptyset^{(\lambda)} \restriction n$ if $X = \emptyset_{(\beta_n)}$.

Define $g(x,\lambda,n)$ by

$$W^X_{g(x,\lambda,n)} = \begin{cases} \emptyset & \text{if } \phi^{\sigma(\lambda,n,X)}_{x,n}(x)\uparrow \\ \mathbb{N} & \text{otherwise.} \end{cases}$$

Suppose that $x \notin \emptyset_{(\lambda)}$. This is true if and only if

$$\phi^{\emptyset^{(\lambda)}}_x(x)\uparrow \iff \forall n \phi^{\emptyset^{(\lambda)}\restriction n}_{x,n}(x)\uparrow \iff \forall n[g(x,\lambda,n) \notin \emptyset_{(\beta_n+1)}].$$

Define $g_l(x,\lambda,n)$ so that $g_l(x,\lambda,n) \notin \emptyset_{(2\beta_n)} \iff g(x,\lambda,n) \notin \emptyset_{(\beta_n+1)}$. (Since β_n is a successor ordinal, $\beta_n + 1 \leq 2\beta_n$.) □

The following lemma contains the heart of the reduction. Given a Π_α fact, we must build a tree of the appropriate limsup rank. Each node of this tree will be associated with a finite set of Π_β assertions for different ordinals β. The behavior of the subtree below a node is as follows. If all the assertions are true, then the rank of the subtree should be large, on the order of the largest β from the set of assertions. But if some Π_β assertion is false, then the rank of the subtree should be small, of a similar height to that β.

The node achieves this behavior by selecting which assertions should be given to each of its child-nodes. The collection of Π_β assertions, if all true, could be viewed as having a generalized Skolem function which covers the first two quantifiers of every assertion in the collection. The previous two lemmas will ensure that this Skolem function, if it exists, is unique. The children try to guess fragments of this unique Skolem function, and each child is given a set of assertions which explore the fragment of the Skolem function that the child provided. The previous two lemmas will ensure that if infinitely many children can correctly guess a fragment of the generalized Skolem function, then (1) all the assertions of the parent are true and (2) these children, having guessed all the right witnesses, will achieve high rank.

On the other hand, if some assertion was false at the level of the parent node, then since the guesses are only fragments, finitely many children will still come up with lucky guesses which give them a pile of true assertions, some of which could be very large compared with the false assertion the parent had. Therefore, the children also each re-evaluate all of the non-maximal assertions from their parent node; this damps the sup of the ranks of the children.

As for damping the limsup, cofinitely many children will automatically dampen down their own ranks through exploring the false assertions generated by their Skolem guesses, which were doomed guesses in a situation where in fact no witnesses existed. Thus the limsup of the ranks of the children is damped. There is a subtlety here. If the limsup is supposed to be damped below some limit ordinal, it is not enough that each child get below that ordinal individually. They have to obey a common bound. That is why, in step (5) below, when α_i is a limit ordinal, M_i is chosen the way it is.

All of the complication that is to follow arises in order to deal with the limit case. When a node is given only one $\Pi_{\alpha+2}$ assertion, each of its children is simply given a single Π_α assertion. If α is finite, the resulting tree has finite height and just one assertion per node. On a first reading it may be helpful to have this special case in mind.

Here is another example, this one for the simplest limit case. If a node is given a single Π_ω assertion, that assertion may be broken up into assertions of size $\Pi_2, \Pi_4, \Pi_6, \Pi_8$, and so on, such that the original assertion is true if and only if all the sub-assertions are true. In that case, most of the children of the node end up totally empty, but of the ones that do not, the first one evaluates only the Π_2 assertion, the second one evaluates the Π_2 and Π_4 assertions, and so on. If all the assertions are true, then the childrens' ranks get bigger the more assertions they evaluate, causing the rank of the whole tree to reach $\omega + 1$. But if the Π_{2n} assertion is false for some n, then every child that evaluates that one has finite rank at most n, and every child that does not evaluate that one has rank at most n as well (because it only evaluates small assertions). So the tree as a whole gets rank at most $n + 1$.

Lemma 4.7. *Let $\alpha_1, \ldots, \alpha_k > 0$ be constructive ordinals, and let $x_1, \ldots, x_k$ be any natural numbers. Recursively in $\alpha_1, \ldots, \alpha_k, x_1, \ldots, x_k$, one may compute a well-founded tree T such that*

- $|T|_{ls} = \max_i \alpha_i + 1$ *if $x_i \notin \emptyset_{(2\alpha_i)}$ for all i*
- $|T|_{ls} \leq \alpha_i$ *whenever $x_i \in \emptyset_{(2\alpha_i)}$.*

Proof. In order to perform the induction we will actually prove something slightly stronger. If $x_i \in \emptyset_{(2\alpha_i)}$ for α_i a limit, given as $\alpha_i = \sup_n \beta_n$, then by Lemma 4.6 there is a least z such that $g_l(x_i, \alpha_i, z) \in \emptyset_{(2\beta_z)}$. In this case, we will ensure that T also satisfies $|T|_{ls} \leq \beta_z + 1$ for that least z.

Define T recursively as follows. Renumber the inputs so that $\alpha_1 \geq \cdots \geq \alpha_k$. (Since all the ordinal notations are comparable, this step is computable). The empty sequence is in T. To compute information about the nth child of the root, decode n as $n = \langle m_0, m_1, \ldots, m_k \rangle$ and do the following:

(1) Check that $m_0 < m_1 < \cdots < m_k$. If it is not, $T_n = \emptyset$.
(2) For any i such that α_i is a limit, check that $m_i = m_{i-1} + 1$. If it does not, then $T_n = \emptyset$.
(3) For any i such that $\alpha_i = 1$, check that $g_0(x_i, m_{i-1}, m_i) = 1$. If it does not, then $T_n = \emptyset$.
(4) If $\alpha_1 = 1$, $T_n = \{\emptyset\}$.
(5) Otherwise, we decide the subtree rooted at $\langle m_0, \ldots, m_k \rangle$ according to membership in the tree which we will now specify. Build a finite set $\mathcal{F}$ of ordinal-input pairs as follows.
   - Let $\mathcal{F}_1 = \{(\alpha_i, x_i) : \alpha_i < \alpha_1\}$
   - Let $\mathcal{F}_2 = \{(\beta, g_s(x_i, m_{i-1}, m_i)) : \alpha_i = \beta + 1 \text{ where } \beta > 0\}$
   - For each limit $\alpha_i = \sup_n \beta_n$, let $M_i \geq m_i$ be least such that for each $(\gamma, x) \in \mathcal{F}_1 \cup \mathcal{F}_2$, if $\gamma < \alpha_i$, then $\gamma \leq \beta_{M_i}$. (Again, this M_i may be effectively computed since the notations involved are all comparable.) For each $n \leq M_i$, let $(\beta_n, g_l(x_i, \alpha_i, n)) \in \mathcal{F}_3$.

   Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. Then T_n is defined recursively as the tree computed from the pairs in $\mathcal{F}$.

This completes the construction.

Observe that the resulting T is well-founded because each time we recurse, the size of the largest ordinal under consideration decreases. Let us verify the properties of this T. We proceed by induction on the size of $\max_i \alpha_i$.

For now on, consider the α_i to be numbered in order, so $\max_i \alpha_i = \alpha_1$.

In the base case, $\alpha_1 = \cdots = \alpha_k = 1$. If $g_0(x_i, m_{i-1}, m_i) = 0$ for any i, then $T = \{\emptyset\}$ and $|T|_{ls} = 1$ which is correct. If $g_0(x_i, m_{i-1}, m_i) = 1$ for all i, step (4) is encountered infinitely often and thus $|T|_{ls} = 2$, which is correct.

Now we consider the case $\alpha_1 > 1$. If, when computing subtree T_n, the algorithm makes it to step 5, then we call n a *recursing child*.

By induction we may always assume that for each child of the root n, $|T_n|_{ls} \leq \alpha_1$. This follows because $|T_n|_{ls} \leq 1$ for non-recursing children n, and for recursing children n, the ordinals considered in order to decide subtree T_n are all less than α_1. Therefore it is always true that $|T|_{ls} \leq \alpha_1 + 1$.

Case 1: The rank should be large. Suppose that for all i, $x_i \notin \emptyset_{(2\alpha_i)}$. Let us see that in this case $|T|_{ls} = \alpha_1 + 1$ is attained. Recall that a child of the root n is decoded as $n = \langle m_0, \ldots, m_k \rangle$. For each choice of m_0, a certain child of the root is obtained by inductively choosing m_i as follows according to the nature of α_i. The functions g_0 and g_s are as defined in Lemma 4.5.

(1) If $\alpha_i = 1$, choose m_i so that $g_0(x_i, m_{i-1}, m_i) = 1$,
(2) If $\alpha_i = \beta + 1$ with $\beta > 0$, choose m_i so that $g_s(x_i, m_{i-1}, m_i) \notin \emptyset_{(2\beta)}$
(3) If α_i is a limit, choose $m_i = m_{i-1} + 1$.

Let n_j be the child so constructed starting with $m_0 = j$. By the definitions of g_0 and g_s, each m_i described above exists, is unique, and satisfies $m_i > m_{i-1}$.

One can check that n_j is a recursing child, and so T_{n_j} is formed using a finite set of ordinal-index pairs (γ, z). Notice that the choices of m_i above, together with the fact that for all i, $x_i \notin \emptyset_{(2\alpha_i)}$, guarantee that $z \notin \emptyset_{(2\gamma)}$ for each of these pairs (γ, z). Therefore, $|T_{n_j}|_{ls}$ will be determined by the largest ordinal under consideration in the construction of T_{n_j}. Now if $a_1 = \beta+1$, then one of the pairs under consideration in the construction of T_{n_j} is $(\beta, g_s(x_1, m_0, m_1))$, and β is maximal among ordinals considered for T_{n_j}. Therefore by the inductive hypothesis, for each j we have $|T_{n_j}|_{ls} = \beta + 1 = \alpha_1$. Since there are infinitely many child subtrees where this rank is obtained, $\limsup_n |T_n|_{ls} = \alpha_1$ and thus $|T|_{ls} = \alpha_1 + 1$ as required. On the other hand, if $\alpha_1 = \sup_n \beta_n$ is a limit, then $(\beta_{M_1}, g_l(x_1, \alpha_1, M_1))$ is used when assembling T_{n_j}, and β_{M_1} is maximal among ordinals considered, because if $\alpha_i < \alpha_1$, then $\beta_{M_1} \geq \alpha_i$, and if $\alpha_i = \alpha_1$, then $M_i = M_1$ (since their selection algorithms are identical). Therefore, by the inductive hypothesis,

$$|T_{n_j}|_{ls} = \beta_{M_1} + 1 > \beta_j + 1$$

because $M_1 \geq m_1 > m_0 = j$. Since $\lim_j \beta_j = \alpha_1$, we have

$$\lim_j |T_{n_j}|_{ls} \geq \lim_j (\beta_j + 1) = \alpha_1$$

as well. Therefore, $\limsup_n |T_n|_{ls} = \alpha_1$ and $|T|_{ls} = \alpha_1 + 1$ as required. Therefore, if for all i, $x_i \notin \emptyset_{(2\alpha_i)}$, then $|T|_{ls} = \alpha_1 + 1$.

Case 2: The rank should be small. On the other hand, suppose that $x_i \in \emptyset_{(2\alpha_i)}$ for some i. Fix an index r at which this occurs. We will show that $|T|_{ls} \leq \alpha_r$.

Subcase 2.1 Suppose $\alpha_r = \beta_r + 1$. By Lemma 4.5 let z_r be such that

$$(\forall z > z_r)(\forall y > z)[g_s(x_r, z, y) \in \emptyset_{(2\beta_r)}]$$

if $\beta_r > 0$, or such that $(\forall z > z_r)(\forall y > z)[g_0(x_r, z, y) = 0]$ if $\beta_r = 0$. One may check that for any child $n = \langle m_0, \ldots, m_k \rangle$ such that $m_{r-1} > z_r$, if n is recursing, then included in consideration for T_n is $(\beta_r, g_s(x_r, m_{r-1}, m_r))$ where $g_s(x_r, m_{r-1}, m_r) \in \emptyset_{(2\beta_r)}$; and if n is not recursing, $T_n = \emptyset$. Therefore by induction, $|T_n|_{ls} \leq \beta_r < \alpha_r$ for such n.

Now let us consider recursing children n such that $m_{r-1} \leq z_r$. There are only finitely many ways $m_0 < \cdots < m_{r-1} \leq z_r$ to begin such children. Fix one such beginning. We claim that for all but at most one choice of the remaining $m_r < \cdots < m_k$, $|T_n|_{ls} < \alpha_r$. That one choice, if it exists, is constructed inductively as in the previous case. That is, for each $i \geq r$, choose m_i to satisfy

(1) If $\alpha_i = 1$, satisfy $g_0(x_i, m_{i-1}, m_i) = 1$,
(2) If $\alpha_i = \beta + 1$ with $\beta > 0$, satisfy $g_s(x_i, m_{i-1}, m_i) \notin \emptyset_{(2\beta)}$, and
(3) If α_i is a limit, let $m_i = m_{i-1} + 1$.

If these m_i exist, they are unique. Suppose we deviate from this recipe in the case of α_i a limit. Then T_n is empty. Suppose we deviate from this one way in the case of $\alpha_i = 1$, and let $g_0(x_i, m_{i-1}, m_i) = 0$. Then by step (3), T_n is empty. Suppose we deviate from this one way in the case of $\alpha_i = \beta + 1$, and include $(\beta, g_s(x_i, m_{i-1}, m_i))$ in the assembling of T_n, where $g_s(x_i, m_{i-1}, m_i) \in \emptyset_{(2\beta)}$. Then by the inductive hypothesis we are guaranteed $|T_n|_{ls} \leq \beta < \alpha_i \leq \alpha_r$. Therefore, considering all children n, there are at most finitely many such that $|T_n|_{ls} \geq \alpha_r$. Therefore, $\limsup_n |T_n|_{ls} \leq \beta_r$.

It remains to show that for each recursing child n, $|T_n|_{ls} \leq \alpha_r$. There are two possibilities. If $\alpha_1 > \alpha_r$, then (α_r, x_r) is included in consideration for T_n, and thus by the inductive hypothesis $|T_n|_{ls} \leq \alpha_r$. On the other hand, if $\alpha_1 = \alpha_r = \beta_r + 1$, then β_r is maximal among ordinals considered for T_n, so by the inductive hypothesis $|T_n|_{ls} \leq \beta_r + 1 = \alpha_r$. Therefore, if α_r is a successor, then $|T|_{ls} \leq \alpha_r$.

Subcase 2.2: Suppose $\alpha_r = \sup_n \beta_n$ is a limit. Using Lemma 4.6, let z_r be least such that $g_{ls}(x_r, \alpha_r, z_r) \in \emptyset_{(2\beta_{z_r})}$. Let us consider children $n = \langle m_0, \dots, m_k\rangle$ such that $m_r \geq z_r$. For each of these n, the pair $(\beta_{z_r}, g_l(x_r, \alpha_r, z_r))$ is used in assembling T_n. So for each such n, $|T_n|_{ls} \leq \beta_{z_r}$.

On the other hand, there are the n such that $m_r < z_r$. There are only finitely many ways $m_0 < \cdots < m_r < z_r$ to begin such an n. We claim that for each such beginning, there is at most one sequence $m_{r+1}, \dots, m_k$ which completes n in such a way that $|T_n|_{ls} > \beta_{z_r}$. The strategy is exactly the same as in the successor case. See (1)-(3) above.

In each case, if such an m_i exists, it is unique. If we deviate from this plan in the case of α_i a limit or $\alpha_i = 1$, then one may check that T_n is empty. If we deviate in the case of $\alpha_i = \beta + 1$ with $\beta > 0$, then we include $(\beta, g_s(x_i, m_{i-1}, m_i)) \in \mathcal{F}_2$, where $g_s(x_i, m_{i-1}, m_i) \in \emptyset_{(2\beta)}$. So to start with, $|T_n|_{ls} \leq \beta$, and if $\beta \leq \beta_{z_r}$ then $|T_n|_{ls}$ is small enough. But if $\beta > \beta_{z_r}$, then this bound is insufficient. In that case, recall that during the creation of $\mathcal{F}_3$ which was used to assemble T_n, we defined M_r to satisfy $M_r \geq m_r$ and $\beta_{M_r} \geq \gamma$ for each $(\gamma, z) \in \mathcal{F}_2$ such that $\gamma < \alpha_r$. Because $\alpha_i < \alpha_r$, $\beta < \alpha_r$. So $\beta_{M_r} \geq \beta > \beta_{z_r}$, so $M_r > z_r$. So in particular, $(\beta_{z_r}, g_l(x_r, \alpha_r, z_r))$ was included when assembling T_n. Therefore, $|T_n|_{ls} \leq \beta_{z_r}$. Therefore, for all but finitely many n, $|T_n|_{ls} \leq \beta_{z_r}$.

It remains to show that for each individual n, $|T_n|_{ls} \leq \beta_{z_r} + 1$.

We now consider two cases. Suppose $\alpha_1 > \alpha_r$. Then for each n, the pair (α_r, x_r) is again under consideration for T_n. But the new leading ordinal is smaller, so by induction, $|T_n|_{ls} \leq \beta_{z_r} + 1$ for each n. On the other hand, if $\alpha_1 = \alpha_r$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_r$, so $M_1 = M_2 = \cdots = M_r$, since the algorithm which selects M_i is the same for each $i = 1, \dots, r$. One may check that β_{M_r} is the largest ordinal under consideration in the assembling of T_n. If $\beta_{M_r} \geq \beta_{z_r}$, then $(\beta_{z_r}, g_l(x_r, \alpha_r, z_r))$ is included, and $|T_n|_{ls} \leq \beta_{z_r}$. On the other hand, if $\beta_{M_r} < \beta_{z_r}$, then since β_{M_r} is largest,

$$|T_n|_{ls} \leq \beta_{M_r} + 1 \leq \beta_{z_r}.$$

Therefore, $\sup_n |T_n|_{ls} \leq \beta_{z_r} + 1$. Therefore, if α_r is a limit with $g_l(x_r, \alpha_r, z_r) \in \emptyset_{(2\beta_{z_r})}$, then $|T|_{ls} \leq \beta_{z_r} + 1 < \alpha_r$. This completes the proof. $\square$

4.4. **Recognizing functions of rank α is $\Pi_{2\alpha+1}$-hard.** In this section we obtain the final result by consideration of what can be encoded into the oscillation sensitivity ε at which a function's rank is witnessed. For this last step, it is necessary to consider functions again instead of trees, because with the trees we only produce functions made of bumps with all the same proportions. In the next theorem, we use functions made of increasingly shallow bumps, and encode the last jump into the uncertainty of how small ε will have to be in order for the bumps which determine the function's rank to be detectable.

Theorem 4.8. *Uniformly in a constructive ordinal $\alpha > 0$ and x, one may find a computable $f \in C[0,1]$ satisfying*

- $x \notin \emptyset_{(2\alpha+1)} \to |f|_{KW} \leq \alpha$
- $x \in \emptyset_{(2\alpha+1)} \to |f|_{KW} = \alpha + 1$

Proof. Given α, x, compute f as follows. Similar to earlier, let $\{[a_n, b_n]\}_{n\in\mathbb{N}}$ be any computable sequence of intervals with rational endpoints satisfying

- Each interval is contained in $(0,1)$
- $b_{n+1} < a_n < b_n$ for each n.
- $\lim_{n\to\infty} a_n = 0$
- $b_n - a_n < a_n^2$

Let g be a computable function satisfying for all x and X,

$$x \in X'' \iff \exists s[g(x,s) \notin X'].$$

Then

$$x \in \emptyset_{(2\alpha+1)} \iff \exists s[g(x,s) \notin \emptyset_{(2\alpha)}].$$

For any s, let $T(s)$ be the tree guaranteed by Lemma 4.7 with input $(\alpha, g(x,s))$. Thus $|T(s)|_{ls} = \alpha+1$ if $g(x,s) \notin \emptyset_{(2\alpha)}$ and $|T(s)|_{ls} \leq \alpha$ otherwise. Then define

$$f = \sum_{s=0}^{\infty} \frac{1}{s+1} f_{T(s)}[a_s, b_s].$$

Recall that Proposition 4.1 guarantees that $||f_{T(s)}|| < 1$, so

$$||\frac{1}{s+1} f_{T(s)}[a_s, b_s]|| < \frac{b_s - a_s}{s+1} < \frac{a_s^2}{s+1}.$$

On neighborhoods bounded away from 0, f is a uniformly presented sum of finitely many computable differentiable functions, but f lives in the envelope of x^2, so it is computable near 0 as well. Thus f is computable and differentiable.

Suppose $x \notin \emptyset_{(2\alpha+1)}$. Then for each s, $g(x,s) \in \emptyset_{(2\alpha)}$, so $|T(s)|_{ls} \leq \alpha$, so $|f_{T(s)}|_{KW} \leq \alpha$. For each $z \neq 0$, there is a neighborhood U of z which intersects exactly one of the $[a_s, b_s]$. Because $P^{\alpha}_{f_{T(s)}[a_s,b_s],\varepsilon} = \emptyset$ for all ε, and $f_{T(s)}[a_s, b_s]$ coincides with f on U, Lemma 4.2 implies that $z \notin P^{\alpha}_{f,\varepsilon}$ for any ε. On the other hand, fix ε and let $z = 0$. Then for any s, by Proposition 4.1, $||f'_{T(s)}|| < 2$, so

$$||\frac{1}{s+1} f'_{T(s)}[a_s, b_s]|| = \frac{1}{s+1}||f'_{T(s)}|| < \frac{2}{s+1}.$$

Let S be large enough that $\frac{4}{S+1} < \varepsilon$. Then for all $p, q, r, s \in [0, b_S)$,

$$\begin{aligned} |\Delta_f(p,q) - \Delta_f(r,s)| &\leq |\Delta_f(p,q)| + |\Delta_f(r,s)| \\ &\leq 2||f' \upharpoonright [0, b_S)|| < \frac{4}{S+1} < \varepsilon, \end{aligned}$$

so $0 \notin P^{\beta}_{f,\varepsilon}$ for any $\beta > 0$. Therefore $P^{\alpha}_{f,\varepsilon} = \emptyset$ for all ε and $|f|_{KW} \leq \alpha$.

On the other hand, suppose that $x \in \emptyset_{(2\alpha+1)}$. Let s be such that $g(x,s) \notin \emptyset_{(2\alpha)}$. Then $T(s)$ has rank $\alpha+1$. So $|f_{T(s)}|_{KW} = \alpha+1$, and this rank is visible at oscillation sensitivity $\varepsilon = \frac{1}{4}$ by Proposition 4.4. So also $|\frac{1}{s+1} f_{T(s)}|_{KW} = |\xi^{-1}(a)|_{\mathcal{O}} + 1$, and this rank is visible at oscillation sensitivity $\varepsilon = \frac{1}{4(s+1)}$. Therefore by Lemmas 4.3 and 4.2,

$$\emptyset \neq P^{\alpha}_{\frac{1}{s+1} f_{T(s)}[a_s,b_s], \frac{1}{4(s+1)}} \subseteq P^{\alpha}_{f, \frac{1}{4(s+1)}}.$$

Thus $|f|_{KW} \geq \alpha+1$. Also, for each s, $|f_{T(s)}|_{KW} \leq \alpha+1$, and $0 \notin P^{\beta}_{f,\varepsilon}$ for any ε and any $\beta > 0$, so just as above, $|f|_{KW} \leq \alpha+1$ always. So in fact $|f|_{KW} = \alpha+1$. $\square$

Therefore, we have the following:

Theorem 4.9. *For each nonzero $\alpha < \omega_1^{CK}$, $D_{\alpha+1}$ is $\Pi_{2\alpha+1}$-complete.*

Proof. By Proposition 3.4, $D_{\alpha+1} \leq_m \overline{\emptyset_{(2\alpha+1)}}$. By Theorem 4.8, $\overline{\emptyset_{(2\alpha+1)}} \leq_m D_{\alpha+1}$. $\square$

Theorem 4.10. *For any limit ordinal $\lambda < \omega_1^{CK}$, D_λ is Σ_λ-complete.*

Proof. First we show that D_λ is Σ_λ. Given $\lambda = \sup_n \beta_n$, we have $e \in D_\lambda \iff \exists n[e \in D_{\beta_n+1}]$. Each $e \in D_{\beta_n+1}$ is $\Pi_{2\beta_n+1}$ by Proposition 3.4, so D_λ is Σ_λ.

Now we show that D_λ is Σ_λ-complete by giving an appropriate reduction. We claim that

$$x \in \emptyset_{(\lambda)} \iff |f_T|_{KW} < \lambda,$$

where T is the tree constructed in Lemma 4.7 from input (λ, x). That lemma guarantees first that $x \notin \emptyset_{(\lambda)}$ implies $|T|_{ls} = \lambda + 1$. Conversely, if $x \in \emptyset_{(\lambda)}$ we have $|T|_{ls} \leq \lambda$. But by Proposition 4.4, the limsup rank of a tree is always a successor, so in fact $x \in \emptyset_{(\lambda)}$ implies $|T|_{ls} < \lambda$. Thus $x \in \emptyset_{(\lambda)} \iff |T|_{ls} < \lambda \iff |f_T|_{KW} < \lambda$. $\square$

References

[AK00] C. J. Ash and J. Knight, *Computable structures and the hyperarithmetical hierarchy*, Studies in Logic and the Foundations of Mathematics, vol. 144, North-Holland Publishing Co., Amsterdam, 2000. MR 1767842 (2001k:03090)

[CR04] Douglas Cenzer and Jeffrey B. Remmel, *Index sets for computable differential equations*, MLQ Math. Log. Q. **50** (2004), no. 4-5, 329–344. MR 2090381 (2005h:03083)

[GMS11] Noam Greenberg, Antonio Montalbán, and Theodore A. Slaman, *The Slaman-Wehner theorem in higher recursion theory*, Proc. Amer. Math. Soc. **139** (2011), no. 5, 1865–1869. MR 2763773 (2012c:03119)

[Grz57] A. Grzegorczyk, *On the definitions of computable real continuous functions*, Fund. Math. **44** (1957), 61–71. MR 0089809 (19,723c)

[Ki97] Haseo Ki, *On the Denjoy rank, the Kechris-Woodin rank and the Zalcwasser rank*, Trans. Amer. Math. Soc. **349** (1997), no. 7, 2845–2870. MR 1390042 (97i:04001)

[KW86] Alexander S. Kechris and W. Hugh Woodin, *Ranks of differentiable functions*, Mathematika **33** (1986), no. 2, 252–278 (1987). MR 882498 (88d:03097)

[Lac55] Daniel Lacombe, *Extension de la notion de fonction récursive aux fonctions d'une ou plusieurs variables réelles. II, III*, C. R. Acad. Sci. Paris **241** (1955), 13–14, 151–153. MR 0072080 (17,225e)

[Maz36] S. Mazurkiewicz, *Über die Menge der differenzierbaren Funktionen.*, Fund. Math. **27** (1936), 244–249.

[Myh71] J. Myhill, *A recursive function, defined on a compact interval and having a continuous derivative that is not recursive*, Michigan Math. J. **18** (1971), 97–98. MR 0280373 (43 #6093)

[PER89] Marian B. Pour-El and J. Ian Richards, *Computability in analysis and physics*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1989. MR 1005942 (90k:03062)

[Sac90] Gerald E. Sacks, *Higher recursion theory*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1990. MR 1080970 (92a:03062)

[Sim09] Stephen G. Simpson, *Subsystems of second order arithmetic*, second ed., Perspectives in Logic, Cambridge University Press, Cambridge, 2009. MR 2517689 (2010e:03073)

[Soa87] Robert I. Soare, *Recursively enumerable sets and degrees*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1987, A study of computable functions and computably generated sets. MR 882921 (88m:03003)

[Spe55] Clifford Spector, *Recursive well-orderings*, J. Symb. Logic **20** (1955), 151–163. MR 0074347 (17,570b)

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